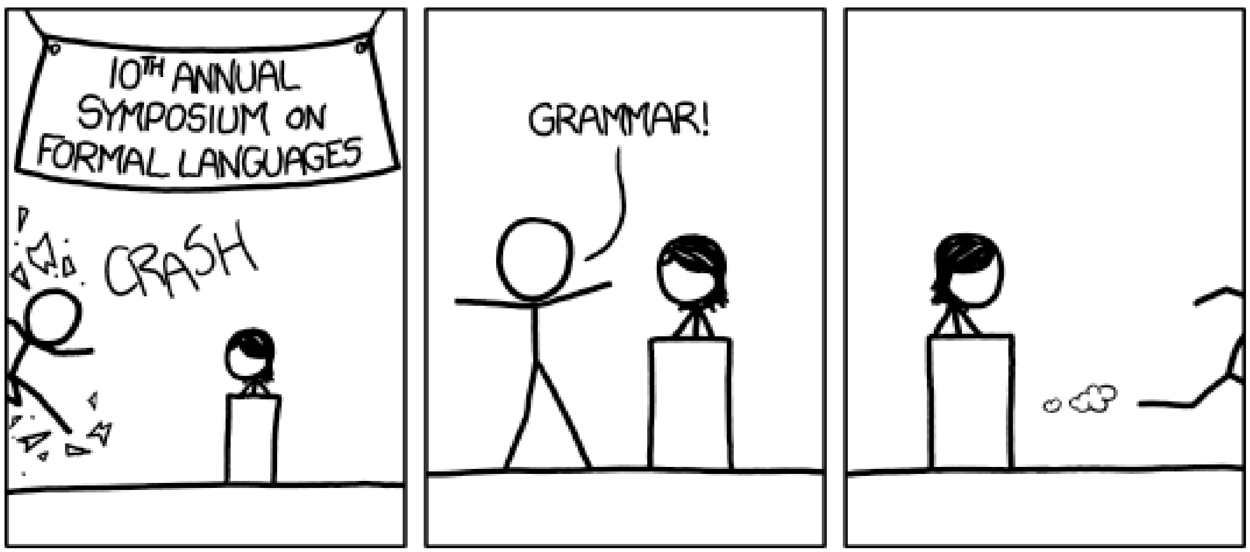
CSE 311: Foundations of Computing

Lecture 19: Context-Free Grammars



[Audience looks around]
“What is going on? There must be some context we’re missing”

Last class: Languages: Sets of Strings

- Subsets of strings are called *languages*
- Examples:
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Binary strings with an equal # of 0's and 1's
  - Legal variable names in Java/C/C++
  - Syntactically correct Java/C/C++ programs
  - Valid English sentences

Last class: Regular Expressions

Regular expressions over Σ

- **Basis**:

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step**:

  If **A** and **B** are regular expressions then so are:

  A ∪ B

  AB

  A*

Last class: Regular Expression is a "pattern"

ε matches the **empty string**

a matches the one character string *a*

A ∪ B matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another

Yields a *language* = the set of strings matched by the regular expression

Last class: Examples

Regular Expression	Language
001^*	{00, 001, 0011, 00111, …}
0^*1^*	{Binary strings with any number of 0s followed by any number of 1s}
$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$	{0000, 1000, 0010, 1010}
$(0^*1^*)^*$	{All binary strings}={0,1}*
$(0 \cup 1)^*$	{All binary strings}={0,1}*
$(0 \cup 1)^*\ 0110\ (0 \cup 1)^*$	{All binary strings containing substring 0110}

Regular Expressions in Practice

- Used to define the *tokens* of a programming language
  - legal variable names, keywords, etc.

- Used in `grep`, a program that does pattern matching searches in UNIX/LINUX

- We can use regular expressions in programs to process strings!

Regular Expressions in Java

```
Pattern p = Pattern.compile("a*b");
Matcher m = p.matcher("aaaaab");
boolean b = m.matches();
```

[01] a 0 or a 1 **^** start of string **$** end of string

[0-9] any single digit **\\.** period **\\,** comma **\\-** minus

. any single character

ab	a followed by b	**(AB)**
(a|b)	a or b	$(\mathbf{A} \cup \mathbf{B})$
a?	zero or one of a	$(\mathbf{A} \cup \varepsilon)$
a*	zero or more of a	$\mathbf{A}^*$
a+	one or more of a	$\mathbf{AA}^*$

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Examples

- All binary strings that have an even # of 1's

Examples

- All binary strings that have an even # of 1's

  e.g., *0* (10*10*)**

Examples

- All binary strings that have an even # of 1's

  e.g., $0^* (10^*10^*)^*$

- All binary strings that *don't* contain 101

Examples

- All binary strings that have an even # of 1's

  e.g., $0^* (10^*10^*)^*$

- All binary strings that *don't* contain 101

  e.g., $0^* (1 \cup 000^*)^* 0^*$

  at least two 0s between 1s

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0's and 1's
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.

Context-Free Grammars

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - Alphabet Σ of *terminal symbols* that can't be replaced
  - A finite set **V** of *variables* that can be replaced
  - One variable, usually **S**, is called the *start symbol*

- The substitution rules involving a variable **A**, written as

$$\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$$

where each w_i is a string of variables and terminals
  - that is $w_i \in (\mathbf{V} \cup \Sigma)^*$

How CFGs generate strings

- Begin with “**S**”
- If there is some variable **A** in the current string, you can replace it by one of the w’s in the rules for **A**
  - $\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$
  - Write this as $x\mathbf{A}y \Rightarrow xwy$
  - Repeat until no variables left
- The set of strings the CFG describes are all strings, containing no variables, that can be *generated* in this manner after a finite number of steps

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

The set of all binary palindromes

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching ***0*1**** but with same number of 0's and 1's)

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Grammar for $\{0^n 1^{2n} : n \geq 0\}$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

S → 0**S**1 | ε

Grammar for $\{0^n 1^{2n} : n \geq 0\}$

S → 0**S**11 | ε

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching ***0*1**** but with same number of 0's and **1**'s)

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

$$\mathbf{S} \rightarrow A10$$

$$\mathbf{A} \rightarrow 0\mathbf{A}1 \mid \varepsilon$$

Example Context-Free Grammars

Example: $S \rightarrow (S) \mid SS \mid \varepsilon$

Example Context-Free Grammars

Example: $S \rightarrow (S) \mid SS \mid \varepsilon$

The set of all strings of matched parentheses

Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s

(not just 0^n1^n, also 0101, 0110, etc.)

Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s

(not just 0^n1^n, also 0101, 0110, etc.)

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$$

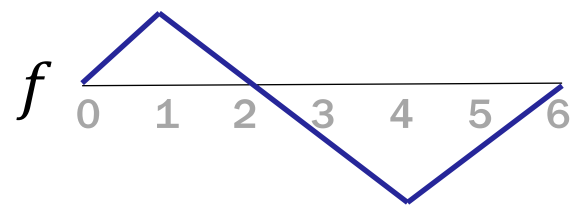
An easy structural induction can show that everything generated by S has an equal # of 0s and 1s

Why does this generate all such strings?

Example Context-Free Grammars

Let $x \in \{0,1\}^*$. Define $f_x(k)$ to be the of 0s minus the number of 1s in the first k characters of x.

E.g., for x = 011100



$f_x(k) = 0$ when first k characters have #0s = #1s

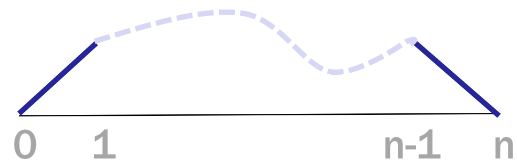
– starts out at 0 $\quad f_x(0) = 0$

– ends at 0 $\quad f_x(n) = 0$

Example Context-Free Grammars

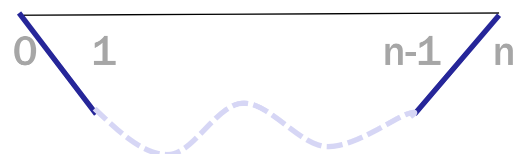
Three possibilities for $f_x(k)$ for $k \in \{1, \ldots, n-1\}$

- $f_x(k) > 0$ for all such k



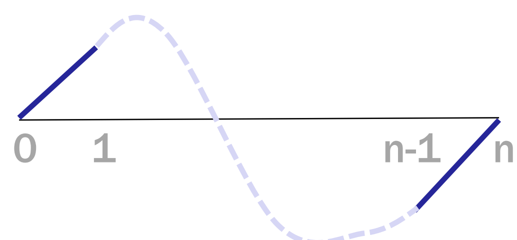
S → 0S1

- $f_x(k) < 0$ for all such k



S → 1S0

- $f_x(k) = 0$ for some such k



S → SS

Simple Arithmetic Expressions

$E \rightarrow$ E+E | E$*$E | (E) | x | y | z | 0 | 1 | 2 | 3 | 4
| 5 | 6 | 7 | 8 | 9

Generate (2$*$x) + y

Simple Arithmetic Expressions

$$E \longrightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate $(2*x) + y$

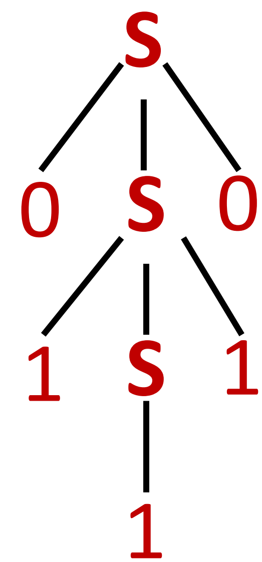
$$E \Rightarrow E+E \Rightarrow (E)+E \Rightarrow (E*E)+E \Rightarrow (2*E)+E \Rightarrow (2*x)+E \Rightarrow (2*x)+y$$

Parse Trees

Suppose that grammar **G** generates a string **x**

- A *parse tree* of **x** for **G** has
  - Root labeled **S** (start symbol of **G**)
  - The children of any node labeled **A** are labeled by symbols of **w** left-to-right for some rule $\mathbf{A} \rightarrow \mathbf{w}$
  - The symbols of **x** label the leaves ordered left-to-right

$$\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$$



Parse tree of 01110

Simple Arithmetic Expressions

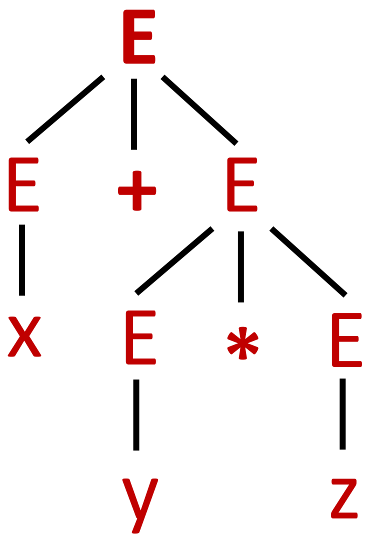
$$E \rightarrow E{+}E \mid E{*}E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate $x+y*z$ in two ways that give two *different* parse trees

Simple Arithmetic Expressions

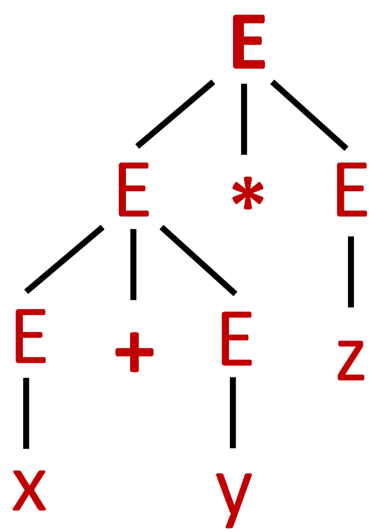
E→ E+E | E∗E | (E) | x | y | z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

Generate x+y∗z in ways that give two *different* parse trees



E ⇒ E+E ⇒ x+E ⇒ x+E*E ⇒ x+y*E ⇒ x+y*z

(add **x** to the product of **y** and **z**)



E ⇒ E*E ⇒ E+E*E ⇒ x+E*E

⇒ x+y*E ⇒ x+y*z

(add **x** to **y**, then multiply by **z**)

building precedence in simple arithmetic expressions

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

$\mathbf{E} \rightarrow \mathbf{T} \mid \mathbf{E+T}$

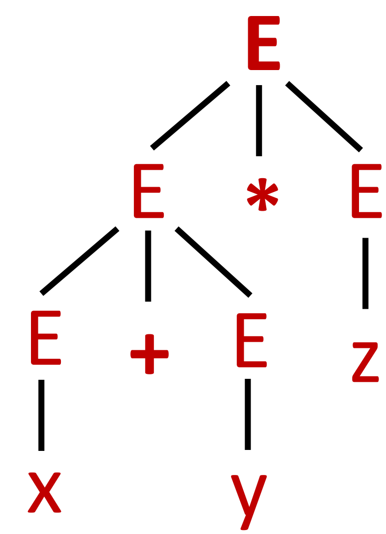
$\mathbf{T} \rightarrow \mathbf{F} \mid \mathbf{F*T}$

$\mathbf{F} \rightarrow (\mathbf{E}) \mid \mathbf{I} \mid \mathbf{N}$

$\mathbf{I} \rightarrow x \mid y \mid z$

$\mathbf{N} \rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$

No longer allows:



building precedence in simple arithmetic expressions

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

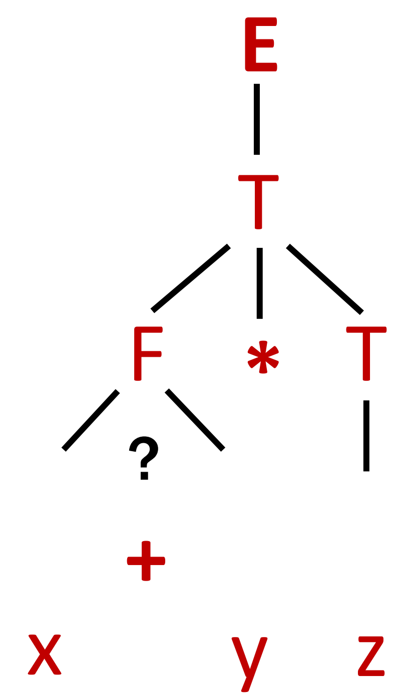
$$\mathbf{E} \rightarrow \mathbf{T} \mid \mathbf{E+T}$$

$$\mathbf{T} \rightarrow \mathbf{F} \mid \mathbf{F} * \mathbf{T}$$

$$\mathbf{F} \rightarrow (\mathbf{E}) \mid \mathbf{I} \mid \mathbf{N}$$

$$\mathbf{I} \rightarrow x \mid y \mid z$$

$$\mathbf{N} \rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$



building precedence in simple arithmetic expressions

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

$$\mathbf{E} \rightarrow \mathbf{T} \mid \mathbf{E{+}T}$$

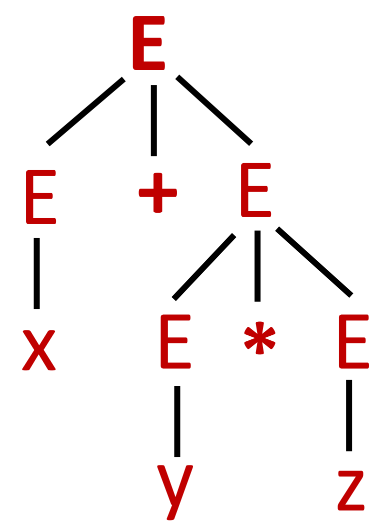
$$\mathbf{T} \rightarrow \mathbf{F} \mid \mathbf{F{*}T}$$

$$\mathbf{F} \rightarrow (\mathbf{E}) \mid \mathbf{I} \mid \mathbf{N}$$

$$\mathbf{I} \rightarrow x \mid y \mid z$$

$$\mathbf{N} \rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Still allows:



building precedence in simple arithmetic expressions

- **E** – expression (start symbol)
- **T** – term   **F** – factor   **I** – identifier **N** - number

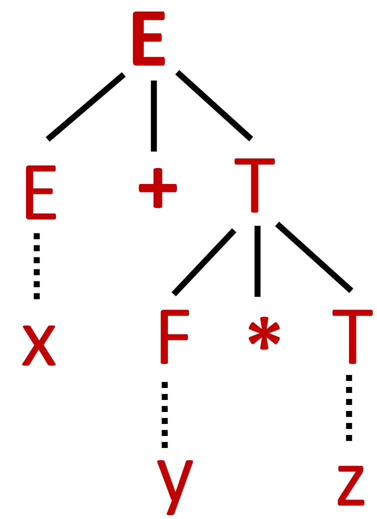
E $\rightarrow$ **T** | **E**+**T**

T $\rightarrow$ **F** | **F**$*$**T**

F $\rightarrow$ (**E**) | **I** | **N**

I $\rightarrow$ x | y | z

N $\rightarrow$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9



CFGs and recursively-defined sets of strings

- A CFG with the start symbol **S** as its *only* variable recursively defines the set of strings of terminals that **S** can generate

- A CFG with more than one variable is a simultaneous recursive definition of the sets of strings generated by *each* of its variables
  – sometimes necessary to use more than one

CFGs and regular expressions

Theorem: For any set of strings (language) A described by a regular expression, there is a CFG that recognizes A.

Proof idea:

P(A) is “A is recognized by some CFG”

Structural induction based on the recursive definition of regular expressions...

Regular Expressions over Σ

- **Basis:**
  - ε is a regular expression
  - ***a*** is a regular expression for any ***a*** $\in \Sigma$
- **Recursive step:**
  - If A and B are regular expressions then so are:

    A $\cup$ B

    AB

    A*

CFGs are more general than REs

- CFG to match RE ε

  S $\rightarrow$ ε

- CFG to match RE ***a*** (for any $a \in \Sigma$)

  S $\rightarrow$ a

CFGs are more general than REs

Suppose CFG with start symbol $\mathbf{S_A}$ matches RE **A**
CFG with start symbol $\mathbf{S_B}$ matches RE **B**

- CFG to match RE $\mathbf{A} \cup \mathbf{B}$

$\mathbf{S} \rightarrow \mathbf{S_A} \mid \mathbf{S_B}$ + rules from original CFGs

- CFG to match RE **AB**

$\mathbf{S} \rightarrow \mathbf{S}_A \mathbf{S}_B$ + rules from original CFGs

CFGs are more general than REs

Suppose CFG with start symbol $\mathbf{S_A}$ matches RE **A**

- CFG to match RE $\mathbf{A}^*$ (= $\varepsilon \cup \mathbf{A} \cup \mathbf{AA} \cup \mathbf{AAA} \cup \ldots$)

  $\mathbf{S} \rightarrow \mathbf{S_A}\,\mathbf{S} \mid \varepsilon$ + rules from CFG with $\mathbf{S_A}$

Backus-Naur Form (The same thing…)

BNF (Backus-Naur Form) grammars

- Originally used to define programming languages
- Variables denoted by long names in angle brackets, e.g.

  <identifier>, <if-then-else-statement>, <assignment-statement>, <condition>

  $::=$ used instead of $\rightarrow$

BNF for C

```
statement:
  ((identifier | "case" constant-expression | "default") ":")*
  (expression? ";" |
   block |
   "if" "(" expression ")" statement |
   "if" "(" expression ")" statement "else" statement |
   "switch" "(" expression ")" statement |
   "while" "(" expression ")" statement |
   "do" statement "while" "(" expression ")" ";" |
   "for" "(" expression? ";" expression? ";" expression? ")" statement |
   "goto" identifier ";" |
   "continue" ";" |
   "break" ";" |
   "return" expression? ";"
  )

block: "{" declaration* statement* "}"

expression:
  assignment-expression%

assignment-expression: (
    unary-expression (
      "=" | "*=" | "/=" | "%=" | "+=" | "-=" | "<<=" | ">>=" | "&=" |
      "^=" | "|="
    )
  )* conditional-expression

conditional-expression:
  logical-OR-expression ( "?" expression ":" conditional-expression )?
```

BNF for (Simple) English

Back to middle school:

<sentence>::=<noun phrase><verb phrase>

<noun phrase>::==<article><adjective><noun>

<verb phrase>::=<verb><adverb>|<verb><object>

<object>::=<noun phrase>

Parse:

The yellow duck squeaked loudly

The red truck hit a parked car