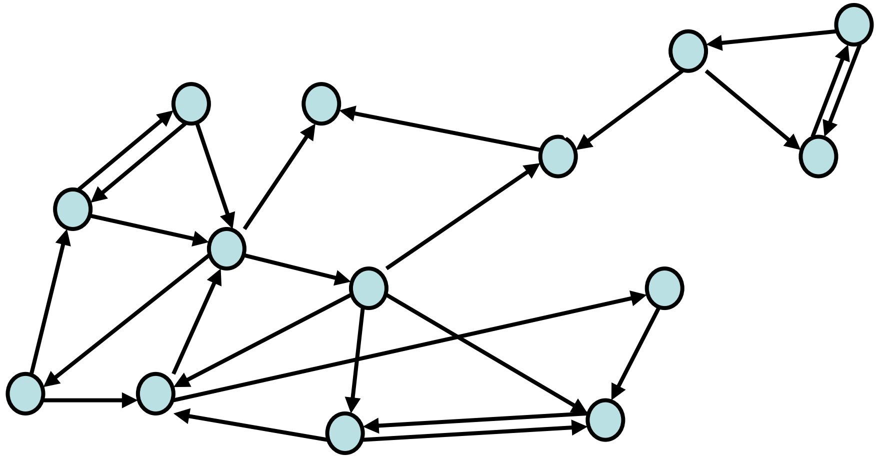
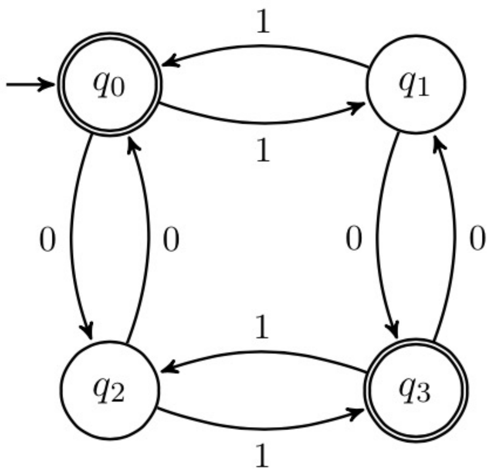
CSE 311: Foundations of Computing

Lecture 21: Directed Graphs, Finite State Machines





Relations

Let A and B be sets,
A **binary relation from** A **to** B is a subset of $A \times B$

(a, b)

$a \in A \quad b \in B$

Let A be a set,
A **binary relation on** A is a subset of $A \times A$

Relations You Already Know

- $\geq$ **on** $\mathbb{N}$ $\subseteq \mathbb{N} \times \mathbb{N}$
  - That is, $\{(x,y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$

$(2,1) \in \geq$

$(1,1) \in \geq$

$(1,2) \notin \geq$

- $<$ **on** $\mathbb{R}$
  - That is, $\{(x,y) : x < y \text{ and } x, y \in \mathbb{R}\}$
- $=$ **on** Σ^*
  - That is, $\{(x,y) : x = y \text{ and } x, y \in \Sigma^*\}$
- $\subseteq$ **on** $\mathcal{P}(U)$ **for universe U**
  - That is, $\{(A,B) : A \subseteq B \text{ and } A, B \in \mathcal{P}(U)\}$

$U = \{1,2,3\}$

$(\emptyset, \{1\}) \in \subseteq$

$(\{1\}, \emptyset) \notin \subseteq$

More Relation Examples

$\mathbf{R_1}$ **=** {(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)}

$\mathbf{R_2}$ **=** {(x, y) : x ≡ y (mod 5) }

$\mathbf{R_3}$ **=** {(c_1, c_2) : c_1 is a prerequisite of c_2 }

$\mathbf{R_4}$ **=** {(s, c) : student s has taken course c }

Properties of Relations

Let R be a relation on A. $\subseteq A \times A$

$\leq$ on $\mathbb{R}$
$\geq$ on $\mathbb{N}$

R is **reflexive** iff $(a,a) \in R$ for every $a \in A$

R is **symmetric** iff $(a,b) \in R$ implies $(b,a) \in R$

$=$ on Σ^*

R is **antisymmetric** iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$

if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

R is **transitive** iff $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

= if $x \geq y$ and $y \geq z$ then $x \geq z$

Which relations have which properties?

$\geq$ **on** $\mathbb{N}$ **:**

$<$ **on** $\mathbb{R}$ **:**

$=$ **on** Σ^* **:**

$\subseteq$ **on** $\mathcal{P}$**(U):**

R_2 = {(x, y) : x ≡ y (mod 5)}:

R_3 = {(c_1, c_2) : c_1 is a prerequisite of c_2 }:

R is **reflexive** iff (a,a) $\in$ R for every a $\in$ A
R is **symmetric** iff (a,b) $\in$ R implies (b, a)$\in$ R
R is **antisymmetric** iff (a,b) $\in$ R and a $\neq$ b implies (b,a) $\notin$ R
R is **transitive** iff (a,b)$\in$ R and (b, c)$\in$ R implies (a, c) $\in$ R

Which relations have which properties?

$\geq$ on $\mathbb{N}$: Reflexive, Antisymmetric, Transitive

$<$ on $\mathbb{R}$: Antisymmetric, Transitive

$=$ on Σ^* : Reflexive, Symmetric, Antisymmetric, Transitive

$\subseteq$ on $\mathcal{P}(U)$: Reflexive, Antisymmetric, Transitive

$R_2 = \{(x, y) : x \equiv y \pmod 5\}$: Reflexive, Symmetric, Transitive

$R_3 = \{(c_1, c_2) : c_1$ is a prerequisite of $c_2\}$: Antisymmetric

R is **reflexive** iff $(a,a) \in R$ for every $a \in A$
R is **symmetric** iff $(a,b) \in R$ implies $(b, a) \in R$
R is **antisymmetric** iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$
R is **transitive** iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Functions

A function $f : A \to B$ (A as input and B as output) is a special type of relation.

A **function f from A to B** is a relation from A to B such that: for every $a \in A$, there is *exactly one* $b \in B$ with $(a, b) \in f$

i.e., for every input $a \in A$, there is one output $b \in B$.
We denote this b by $f(a)$.

Function composition: If $f : A \to B$ and $g : B \to C$ then their **composition** $g \circ f : A \to C$ is defined by

$$g \circ f\,(a) = g(f(a))$$

Composing Relations

Let R be a relation from A to B. $\subseteq A \times B$
Let S be a relation from B to C. $\subseteq B \times C$

The composition of R and S, $S \circ R$ is the relation from A to C defined by:

$$S \circ R = \{(a, c) : \exists \text{ b such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

Intuitively, a pair is in the composition if there is a "connection" from the first to the second.

The order of writing composition generalizes the function case

Examples

$f(g(x))$ $g(f(x))$

(a,b) ∈ Parent iff b is a parent of a

(a,b) ∈ Sister iff b is a sister of a

When is (x,y) ∈ Sister ∘ Parent? aunt

When is (x,y) ∈ Parent ∘ Sister?

$S(R(a))$

S ∘ R = {(a, c) | ∃ b such that (a,b) ∈ R and (b,c) ∈ S}

Powers of a Relation

$$R^2 = R \circ R$$
$$= \{(a, c) : \exists b \text{ such that } (a, b) \in R \text{ and } (b, c) \in R\}$$

$R^0 = \{(a, a) : a \in A\}$ "the equality relation on A"

$R^{n+1} = R^n \circ R$ for $n \geq 0$

$R \circ R^n$

$R^n = \underbrace{R \circ \cdots \circ R}_{n \text{ times}}$

e.g., $R^1 = R^0 \circ R = R$

$R^2 = R^1 \circ R = R \circ R$

Matrix Representation

Relation $\boldsymbol{R}$ on $\boldsymbol{A} = \{a_1, \dots, a_n\}$

$$\boldsymbol{m_{ij}} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \boldsymbol{R} \\ 0 & \text{if } (a_i, a_j) \notin \boldsymbol{R} \end{cases}$$

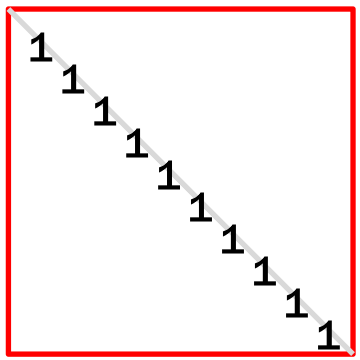
{ (1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3) }

	1	**2**	**3**	**4**
1	1	1	0	1
2	1	0	1	0
3	0	1	1	0
4	0	1	1	0

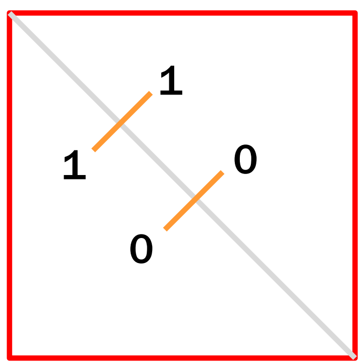
Properties using matrix representation

$(a,a) \in R$ for all $a \in A$

reflexive

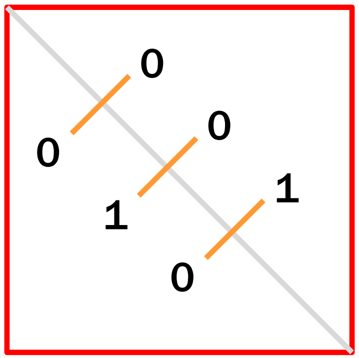


symmetric



Same when rows & columns swapped

anti-symmetric

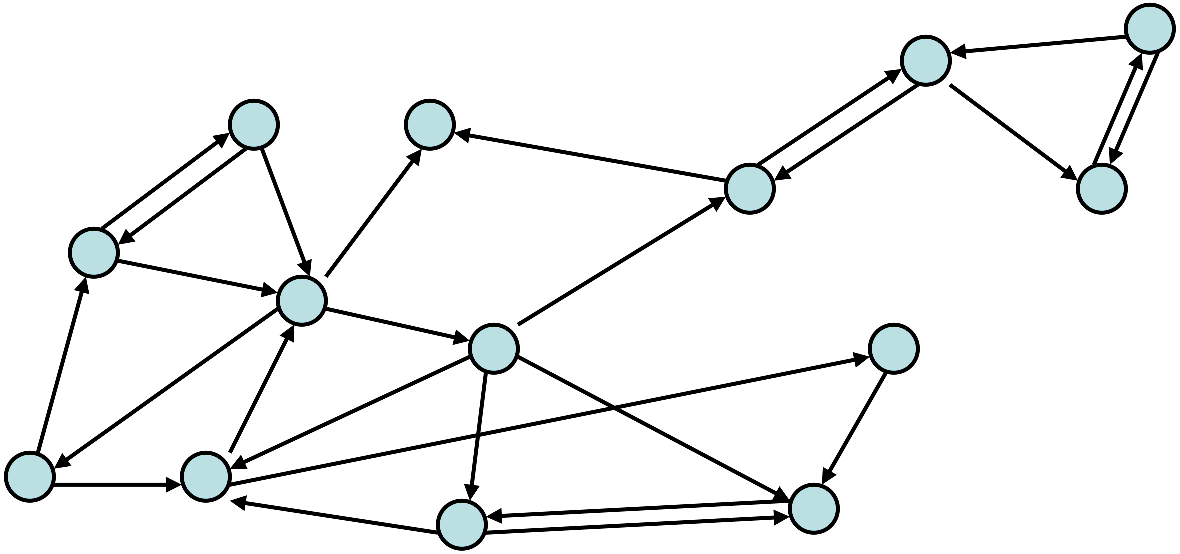


No 1-1 pairs

Directed Graphs

G = (V, E)

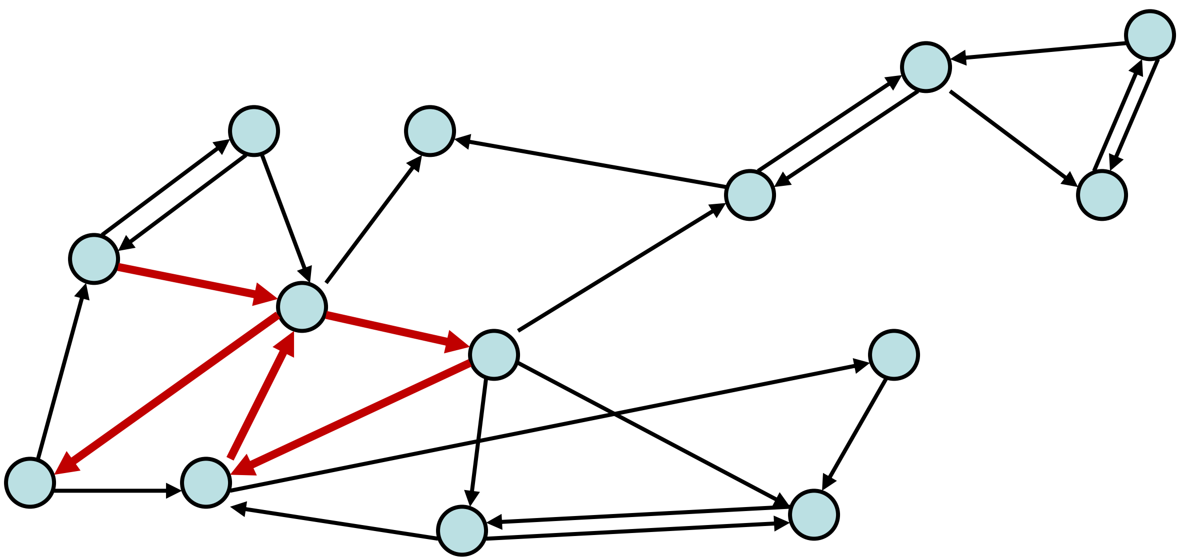
V – vertices
E – edges, ordered pairs of vertices



Directed Graphs

G = (V, E)   V – vertices
E – edges   (relation on vertices)

Path: $v_0, v_1, \ldots, v_k$ with each (v_i, v_{i+1}) in E

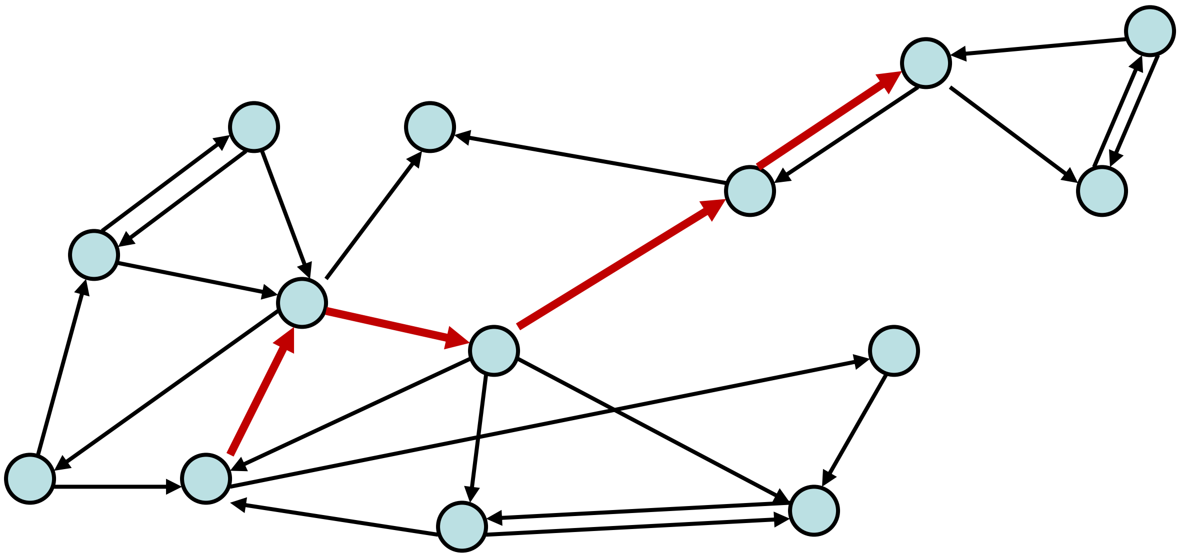


Directed Graphs

G = (V, E) V – vertices
E – edges (relation on vertices)

Path: $v_0, v_1, \ldots, v_k$ with each (v_i, v_{i+1}) in E

Simple Path: none of $\mathbf{v_0}, \ldots, \mathbf{v_k}$ repeated
Cycle: $\mathbf{v_0 = v_k}$
Simple Cycle: $\mathbf{v_0 = v_k}$, none of $\mathbf{v_1}, \ldots, \mathbf{v_k}$ repeated



Directed Graphs

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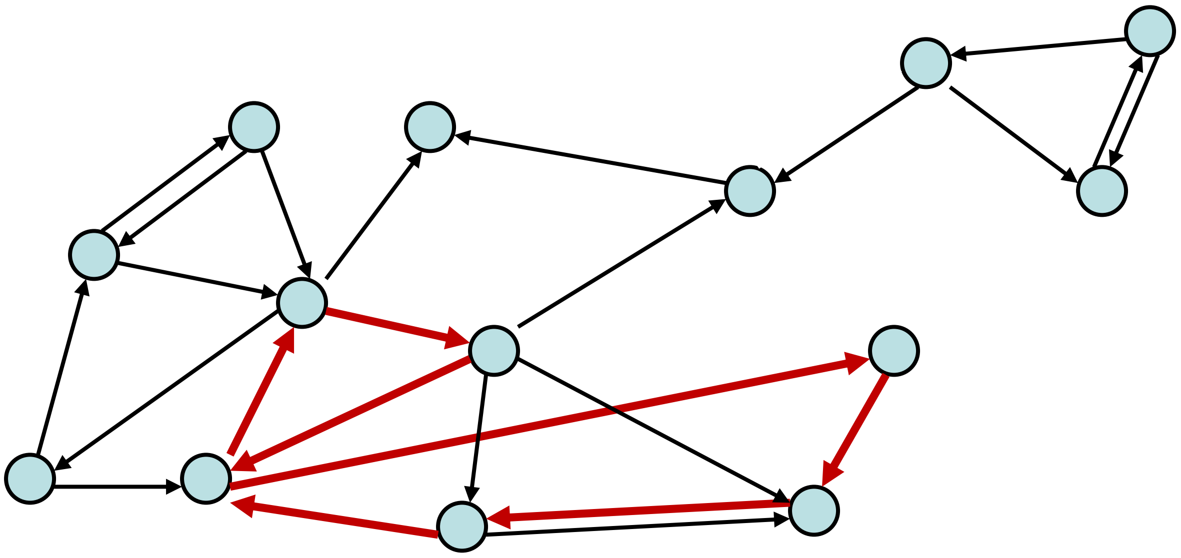

Directed Graphs

G = (V, E) V – vertices
E – edges (relation on vertices)

Path: $v_0, v_1, \ldots, v_k$ with each (v_i, v_{i+1}) in E

Simple Path: none of $\mathbf{v_0}, \ldots, \mathbf{v_k}$ repeated
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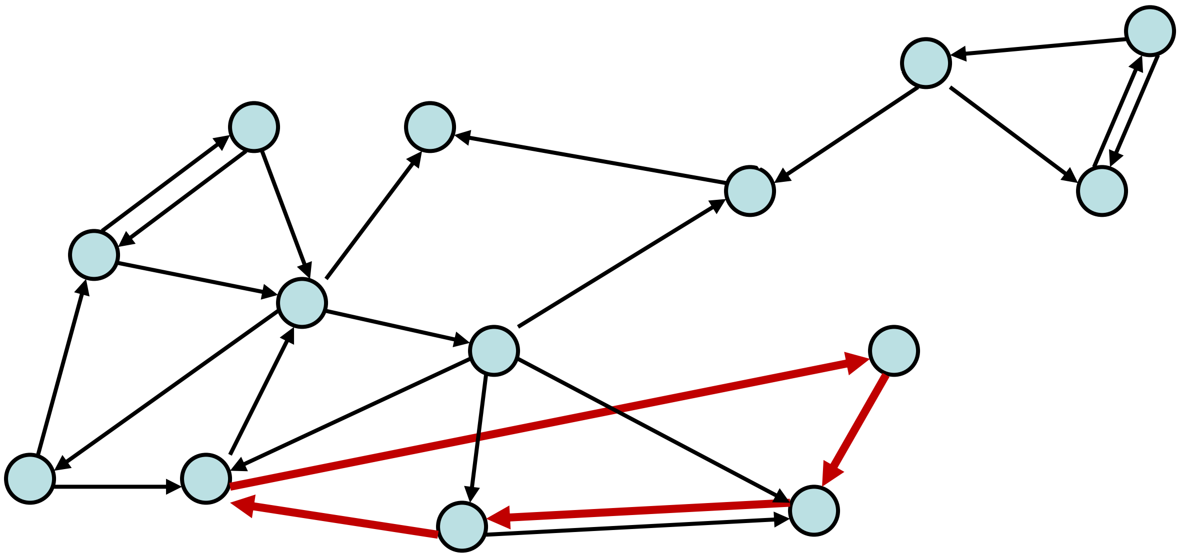
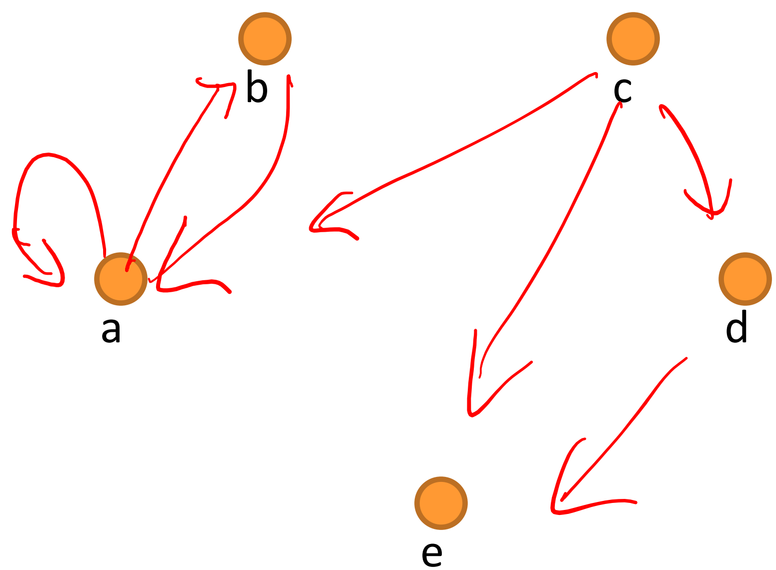


Representation of Relations

Directed Graph Representation (Digraph)

V = {a, b, c, d, e}

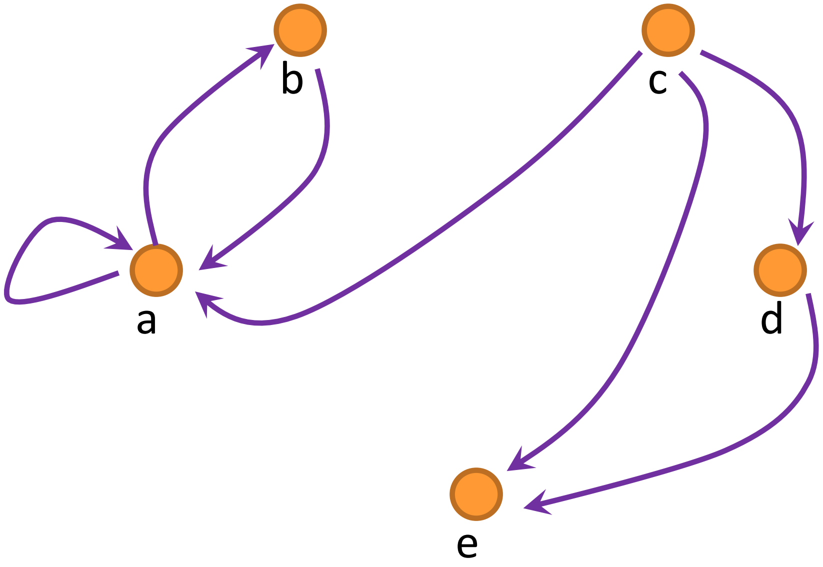
E = {(a, b), (a, a), (b, a), (c, a), (c, d), (c, e), (d, e) }



Representation of Relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



Relational Composition using Digraphs

If $\boldsymbol{S} = \{(2,2),(2,3),(3,1)\}$ and $\boldsymbol{R} = \{(1,2),(2,1),(1,3)\}$

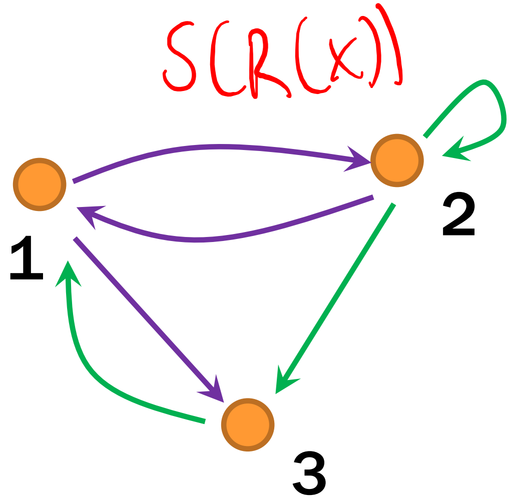
Compute $\boldsymbol{S} \circ \boldsymbol{R}$

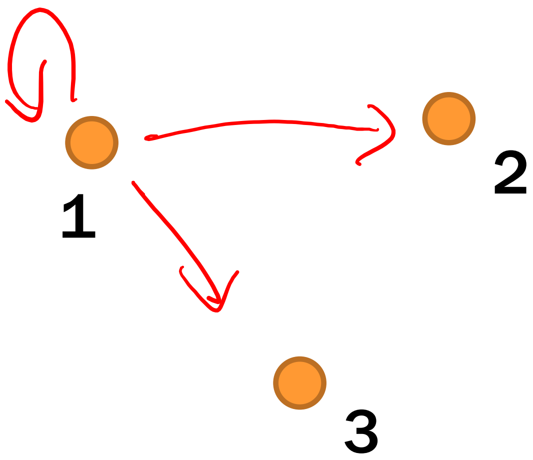
$A = \{1, 2, 3\}$

Relational Composition using Digraphs

If $S = \{(2,2),(2,3),(3,1)\}$ and $R = \{(1,2),(2,1),(1,3)\}$

Compute $S \circ R$





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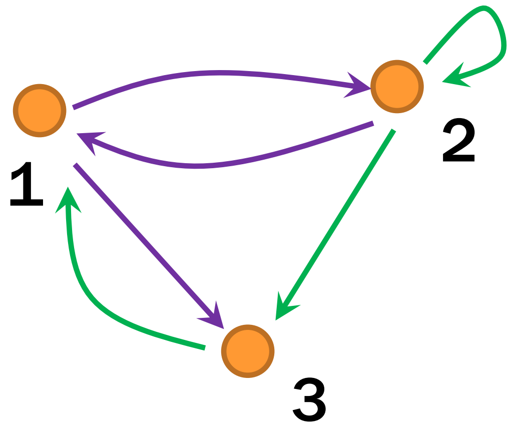

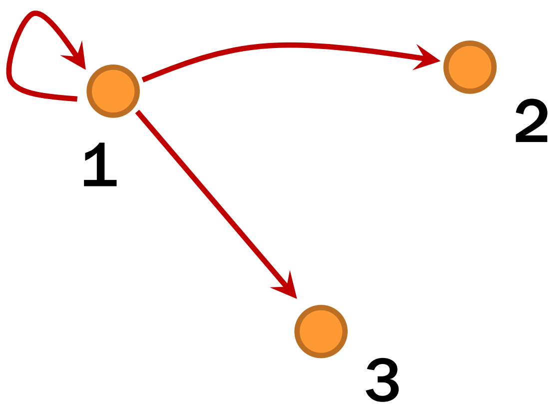

Relational Composition using Digraphs

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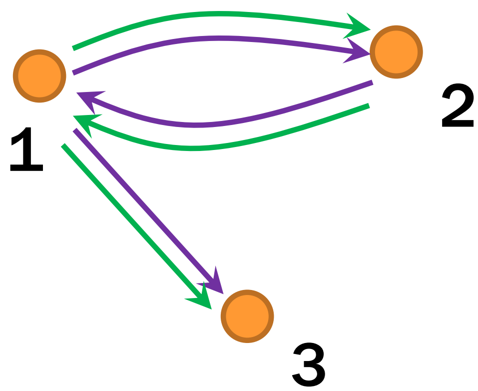
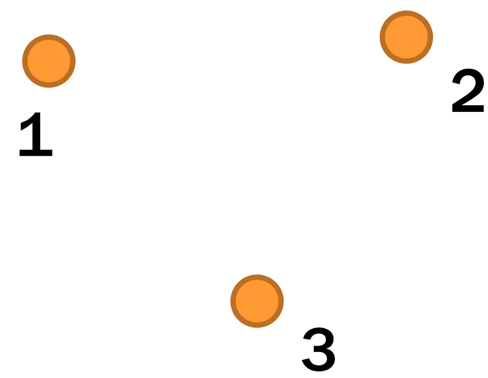




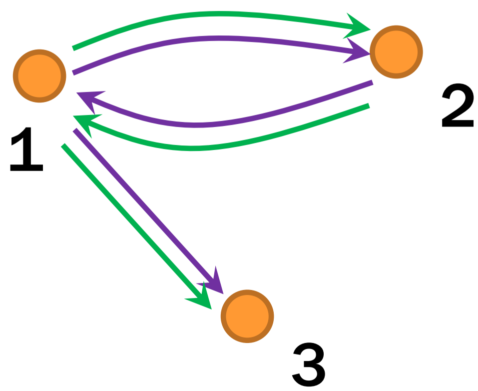
$(a, c) \in R \circ R = R^2$ iff $\exists b\ ((a, b) \in R \land (b, c) \in R)$

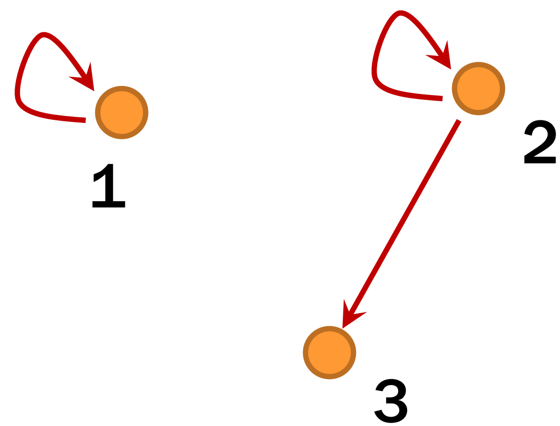
iff $\exists b$ such that a, b, c is a path

Relational Composition using Digraphs

If $\boldsymbol{R} = \{(1,2),(2,1),(1,3)\}$ and $\boldsymbol{R} = \{(1,2),(2,1),(1,3)\}$

Compute $\boldsymbol{R} \circ \boldsymbol{R}$





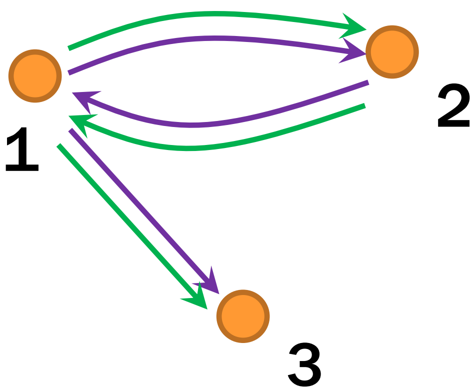
$(a, c) \in R \circ R = R^2$ iff $\exists b\ ((a, b) \in R \land (b, c) \in R)$

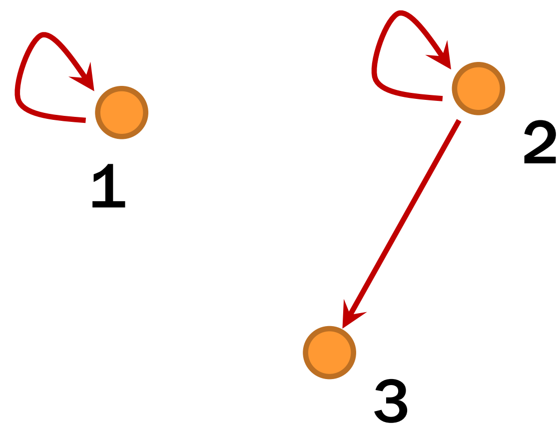
iff $\exists b$ such that a, b, c is a path

Relational Composition using Digraphs

If $R = \{(1,2),(2,1),(1,3)\}$ and $R = \{(1,2),(2,1),(1,3)\}$

Compute $R \circ R$





Special case: $R \circ R$ is paths of length 2.

- R is paths of length 1
- R^0 is paths of length 0 (can't go anywhere)
- $R^3 = R^2 \circ R$ etc, so is R^n paths of length n

Paths in Graphs and Relations

Def: The **length** of a path in a graph is the number of edges in it (counting repetitions if edge used > once).

Elements of R^0 correspond to paths of length 0.
Elements of $R^1 = R$ are paths of length 1.
Elements of R^2 are paths of length 2.
…

Paths in Graphs and Relations

Def: The **length** of a path in a graph is the number of edges in it (counting repetitions if edge used > once).

Let $\boldsymbol{R}$ be a relation on a set $\boldsymbol{A}$.

There is a path of length $\boldsymbol{n}$ from **a** to **b** in the digraph for $\boldsymbol{R}$ if and only if (**a,b**) $\in \boldsymbol{R^n}$

Connectivity In Graphs

Def: Two vertices in a graph are **connected** iff there is a path between them.

Let $\boldsymbol{R}$ be a relation on a set $\boldsymbol{A}$. The **connectivity** relation $\boldsymbol{R}^*$ consists of the pairs (a, b) such that there is a path from a to b in $\boldsymbol{R}$.

$$R^* = \bigcup_{k=0}^{\infty} R^k$$

Note: The Rosen book uses the wrong definition of this quantity. What the Rosen defines (ignoring $k = 0$) is usually called R+

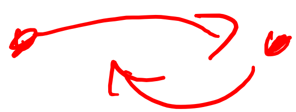
How Properties of Relations show up in Graphs

Let R be a relation on A.

R is **reflexive** iff (a,a) ∈ R for every a ∈ A

for each a ∈ A

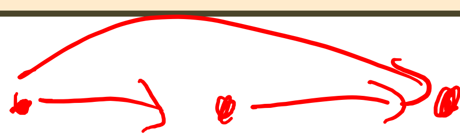
R is **symmetric** iff (a,b) ∈ R implies (b, a)∈ R



R is **antisymmetric** iff (a,b) ∈ R and a ≠ b implies (b,a) ∉ R

for any a, b ∈ A s.t. a ≠ b

R is **transitive** iff (a,b)∈ R and (b, c)∈ R implies (a, c) ∈ R



How Properties of Relations show up in Graphs

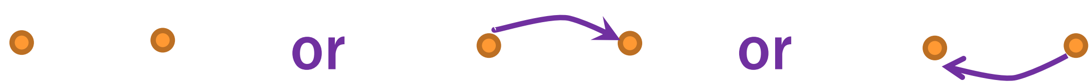
Let R be a relation on A.

R is **reflexive** iff $(a,a) \in R$ for every $a \in A$

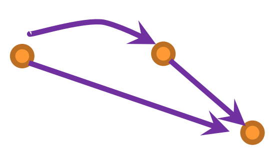
at every node

R is **symmetric** iff $(a,b) \in R$ implies $(b, a) \in R$

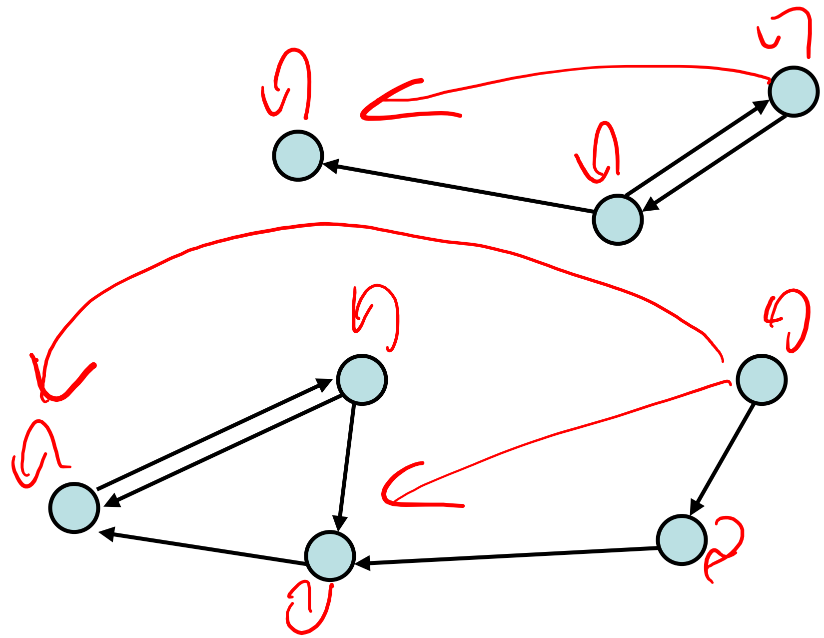
R is **antisymmetric** iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$



R is **transitive** iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

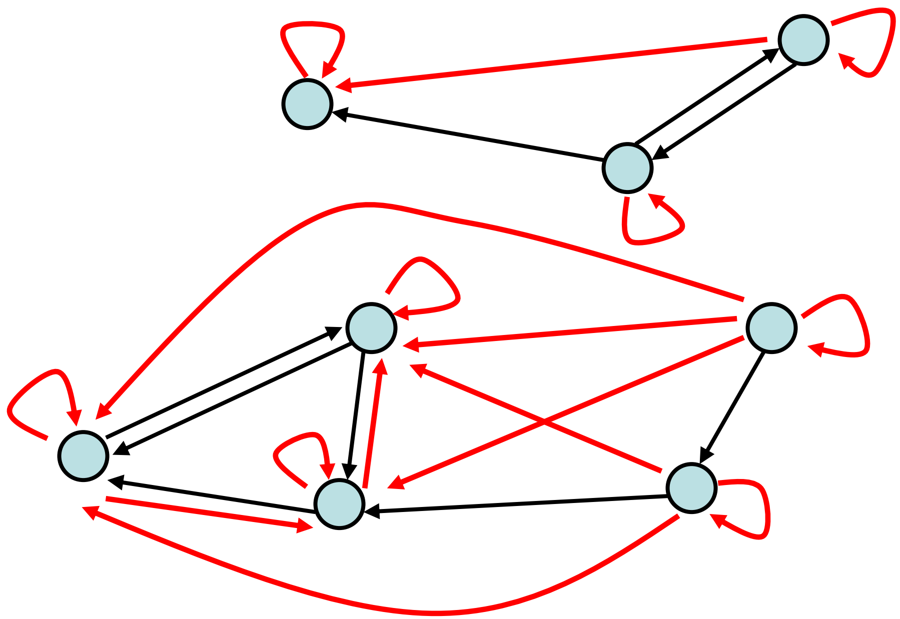


Transitive-Reflexive Closure



Add the **minimum possible** number of edges to make the relation transitive and reflexive.

Transitive-Reflexive Closure



Relation with the **minimum possible** number of **extra edges** to make the relation both transitive and reflexive.

The **transitive-reflexive closure** of a relation $\boldsymbol{R}$ is the connectivity relation $\boldsymbol{R}^*$

n-ary Relations

$\subseteq A \times B$

Let $\boldsymbol{A_1, A_2, \ldots, A_n}$ be sets. An $\boldsymbol{n}$**-ary** relation on these sets is a subset of $\boldsymbol{A_1 \times A_2 \times \cdots \times A_n}$.

Relational Databases

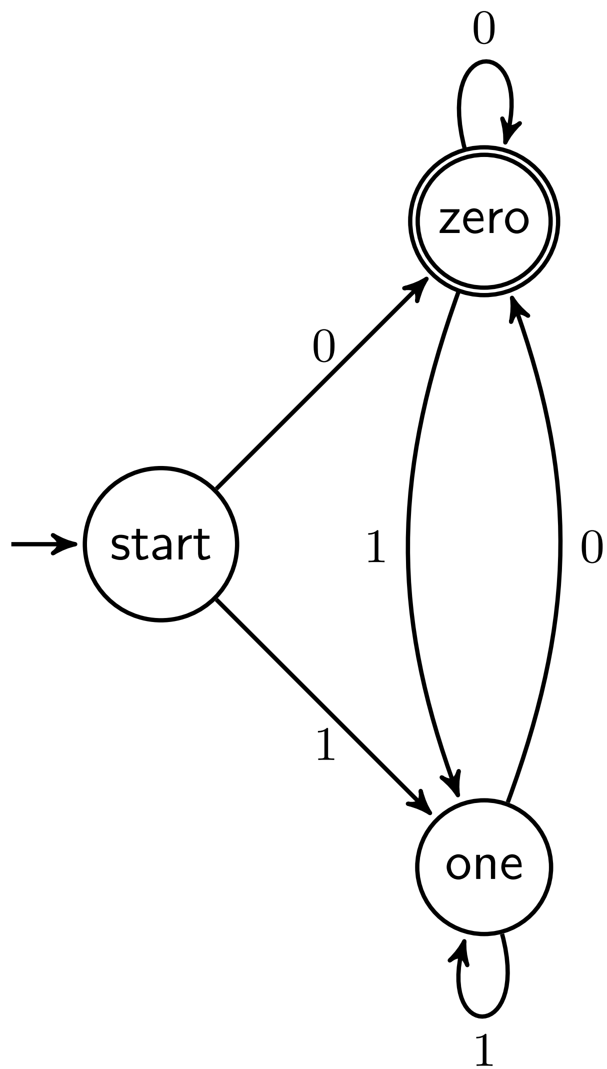
STUDENT

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Von Neuman	481080220	555	3.78
Russell	238082388	022	3.85
Einstein	238001920	022	2.11
Newton	1727017	333	3.61
Karp	348882811	022	3.98
Bernoulli	2921938	022	3.21

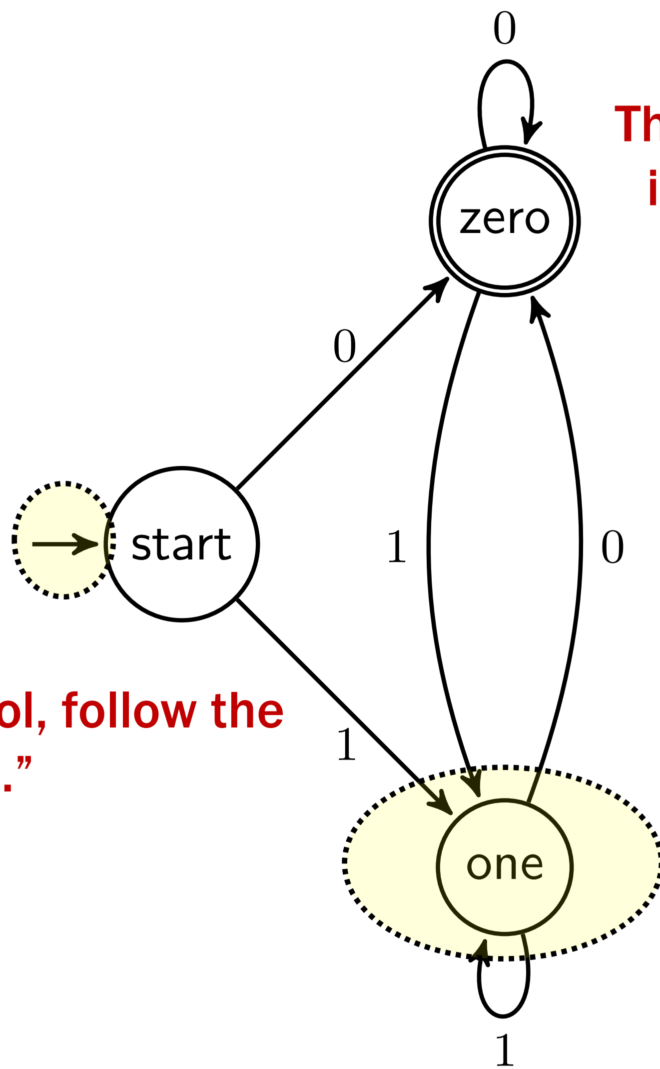
Back to Languages

Selecting strings using labeled graphs as “machines”

$\Sigma = \{0,1\}$



Finite State Machines



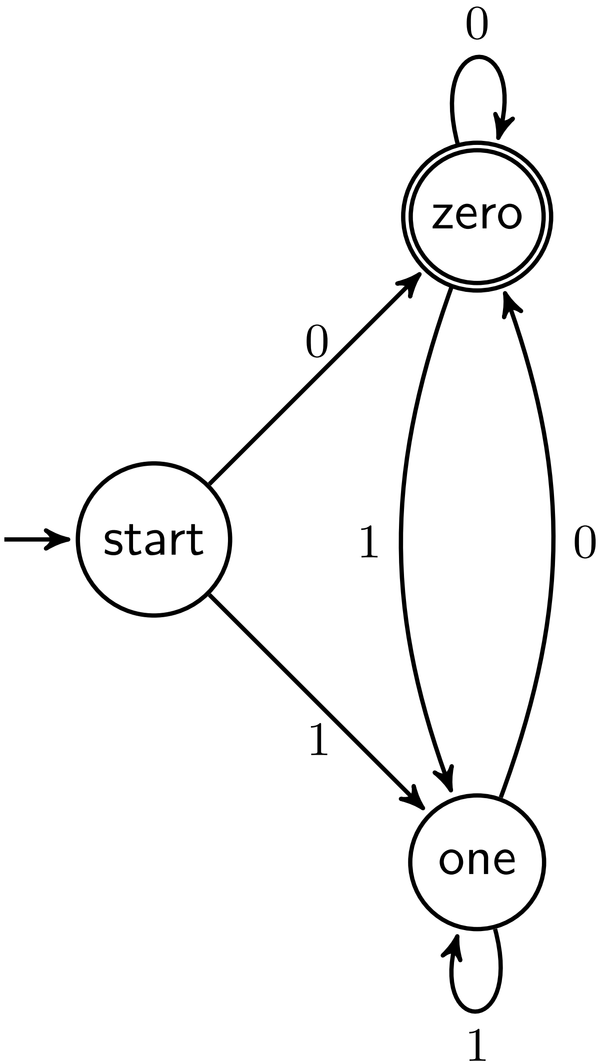
The "double circle" means "the input is good if it ends here"

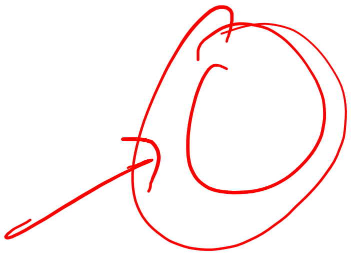
"Start here"

"If I get this symbol, follow the arrow…"

The circles are called "states"
We're only in a single state at any point in time…

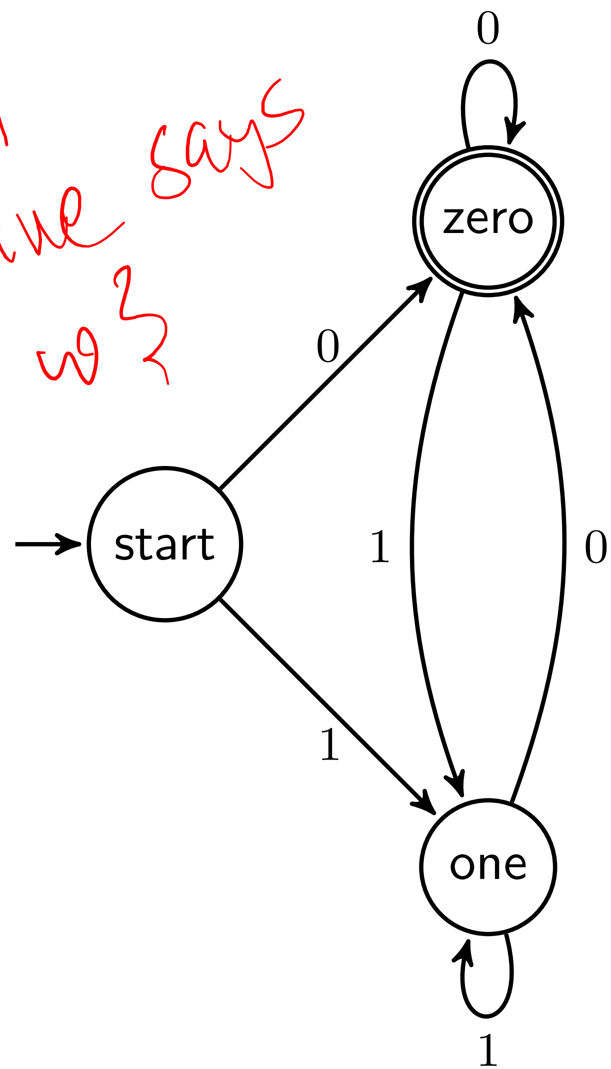
Which strings does this machine say are OK?





Which strings does this machine say are OK?

$\{w \in \Sigma^* \mid \text{this machine says yes to } w\} \subseteq \Sigma^*$



The set of all binary strings that end in 0