# Quiz Section 6: Ordinary, Strong, and Structural Induction

## Task 1 – Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- soy($x$) is true iff $x$ contains soy milk.
- whole($x$) is true iff $x$ contains whole milk.
- sugar($x$) is true iff $x$ contains sugar
- decaf($x$) is true iff $x$ is not caffeinated.
- vegan($x$) is true iff $x$ is vegan.
- RobbieLikes($x$) is true iff Robbie likes the drink $x$.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

**a)** Coffee drinks with whole milk are not vegan.

**b)** Robbie only likes one coffee drink, and that drink is not vegan.

**c)** There is a drink that has both sugar and soy milk.

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \wedge \texttt{RobbieLikes}(x)] \rightarrow \texttt{sugar}(x))$$

## Task 2 – Casting Out Nines

Let $m \in \mathbb{N}$. This problem proves that if $9|m$, then the sum of the digits of $m$ is a multiple of 9. (It actually proves a bit more.) In order to state this one needs the base 10 representation of $m$. Write $m = (d_n d_{n-1} \cdots d_1 d_0)_{10}$ where $d_0, \ldots, d_n$ are the base-10 digits of $m$; that is, each $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$ and $m = \sum_{i=0}^{n} d_i 10^i$.

Prove that casting out nines works for all $m \in \mathbb{N}$ by induction on the number of digits of $m$ by showing that $m$ and the sum of its digits are equivalent modulo 9. We can write this using summation notation as: Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} d_i 10^i \equiv \sum_{i=0}^{n} d_i \pmod 9$ for all $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$. (In other words, prove that for all $n \in \mathbb{N}$, for all $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$,

$$d_0 + 10^1 \cdot d_1 + 10^2 \cdot d_2 + \cdots + 10^n \cdot d_n \equiv d_0 + d_1 + d_2 + \cdots + d_n \pmod 9.$$

## Task 3 – In Harmony with Ordinary Induction

Define

$$H_i = \sum_{j=1}^{i} \frac{1}{j} = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

The numbers $H_i$ are called the *harmonic* numbers.
Prove that $H_{2^n} \geqslant 1 + \frac{n}{2}$ for all integers $n \geqslant 0$.

## Task 4 – Induction with Formulas

These problems are a little more abstract.

**a)** i. Show that given two sets $A$ and $B$ that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

ii. Show using induction that for an integer $n \geqslant 2$, given $n$ sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

**b)** i. Show that given any integers $a$, $b$, and $c$, if $c \mid a$ and $c \mid b$, then $c \mid (a+b)$. (Don't use induction.)

ii. Show using induction that for any integer $n \geqslant 2$, given $n$ numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer $c$ such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

## Task 5 – Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in year $n$ is described by the function $f(n)$:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geqslant 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$. That is, construct a formula for $f(n)$ and prove its correctness.

## Task 6 – Strong Induction

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all $n \geqslant 1$.

## Task 7 – Structural Induction

Define the set $S$ as follows:
**Basis Step:** $[1,1,0] \in S$ and $[0,1,1] \in S$.
**Recursive Step:** If $[u,v,w] \in S$ and $[u',v',w'] \in S$
and $\alpha \in \mathbb{R}$ then $[\alpha u, \alpha v, \alpha w] \in S$ and $[u+u', v+v', w+w'] \in S$.

Prove that every $x \in S$ can be written in the form $x = [u,v,w]$ where $u,v,w \in \mathbb{R}$ and $v = u + w$.