# Strong Induction

CSE 311 Winter 2024
Lecture 13

# How do we know recursion works?

```
//Assume i is a nonnegative integer
//returns 2^i.
public int CalculatesTwoToTheI(int i){
     if(i == 0)
          return 1;
     else
          return 2*CaclulatesTwoToTheI(i-1);
}
```

Why does `CalculatesTwoToTheI(4)` calculate `2^4?`

Convince the other people in your room

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on $n$.
2. Base Case: Show $P(0)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n$ by the principle of induction.

# More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

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Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$."

We show $P(n)$ holds for all $n$ by induction on $n$.

Base Case ( )

Inductive Hypothesis:

Inductive Step:

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

# More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$."

We show $P(n)$ holds for all natural numbers $n$ by induction on $n$.

Base Case $(n = 0)$ $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: We show $P(k + 1)$. Consider the summation $\sum_{i=0}^{k+1} 2^i = 2^{\mathrm{k}+1} + \sum_{i=0}^{k} 2^i = 2^{k+1} + 2^{k+1} - 1$, where the last step is by IH.

Simplifying, we get: $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1 = 2^{(k+1)+1} - 1$.

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

# Let's Try Another Induction Proof

Let $g(n) = \begin{cases} 2 & \text{if } n = 2 \\ g(n-1)^2 + 3g(n-1) & \text{if } n > 2 \end{cases}$

Prove $g(n)$ is even for all $n \geq 2$ by induction on $n$.

Let's just set this one up, we'll leave the individual pieces as exercises.

# Setup

Let $P(n)$ be "$g(n)$ is even."

HEY WAIT -- $P(0)$ isn't true $g(0)$ isn't even defined!

We can move the "starting line"

Change the base case, and then update the IH to have the smallest value of $k$ assume just the base case.

# Setup

Let $P(n)$ be "$g(n)$ is even."

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

**Base Case ($\boldsymbol{n = 2}$):** <todo>

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $\mathrm{k} \geq 2$.

**Inductive Step:**

<todo>

We conclude $P(k + 1)$. Therefore, $P(n)$ holds for all $n \geq 2$ by the principle of induction.

# Setup

Let $P(n)$ be "$g(n)$ is even."

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

**Base Case** $(\boldsymbol{n = 2})$: $g(n) = 2$ by definition. 2 is even, so we have $P(2)$.

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $\mathrm{k} \geq 2$.

**Inductive Step:** We show $P(k+1)$. Consider $g(k+1)$. By definition of $g(\cdot)$, $g(k+1) = g(k)^2 + 3g(k)$. By inductive hypothesis, $g(k)$ is even, so it equals $2j$ for some integer $j$. Plugging in we have:

$g(k+1) = (2j)^2 + 3(2j) = 2\left(2j^2\right) + 2(3j) = 2(2j^2 + 3j)$.

Since $j$ is an integer, $2j^2 + 3j$ is also an integer, and $g(k+1)$ is even.

Therefore, $P(n)$ holds for all $n \geq 2$ by the principle of induction.

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on $n$.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

# Let’s Try Another Induction Proof

**Fundamental Theorem of Arithmetic**

**Every positive integer greater than 1 has a unique prime factorization.**

Uniqueness is hard. Let’s just show existence.

I.e.

Claim: Every positive integer greater than 1 can be written as a product of primes.

# Prime Factorizations

Some examples

$12 = 2^2 \cdot 3$

$35 = 5 \cdot 7$

$36 = 2^2 \cdot 3^2$

$7 = 7$

Notice, for prime numbers the product is just the one number.

# Induction on Primes.

Let $P(n)$ be "$n$ can be written as a product of primes."

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

**Base Case ($\boldsymbol{n = 2}$):** 2 is a product of just itself. Since $2$ is prime, it is written as a product of primes.

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary integer $k \geq 2$.

**Inductive Step:**
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

# We're Stuck

We can divide $k+1$ up into smaller pieces (say $s, t$ such that $st = k+1$ with $2 \leq s < k+1$ and $2 \leq t < k+1$

Is $P(s)$ true? Is $P(t)$ true?

I mean…it would be…

But in the inductive step we don't have it…

Let's add it to our inductive hypothesis.

# Induction on Primes

Let $P(n)$ be "$n$ can be written as a product of primes."

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

**Base Case ($\boldsymbol{n = 2}$):** 2 is a product of just itself. Since $2$ is prime, it is written as a product of primes.

**Inductive Hypothesis:**

**Inductive Step:**
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

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Therefore $P(k + 1)$.

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# Induction on Primes

Let $P(n)$ be "$n$ can be written as a product of primes."

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

**Base Case ($n = 2$):** 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

**Inductive Hypothesis:** Suppose $P(2), \ldots, P(k)$ hold for an arbitrary integer $k \geq 2$.

**Inductive Step:**
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite: We can write $k + 1 = st$ for $s, t$ nontrivial divisors (i.e. $2 \leq s < k + 1$ and $2 \leq t < k + 1$). By inductive hypothesis, we can write $s$ as a product of primes $p_1 \cdot \ldots p_j$ and $t$ as a product of primes $q_1 \cdots q_\ell$. Multiplying these representations, $k + 1 = p_1 \cdots p_j \cdot q_1 \cdots q_\ell$, which is a product of primes.

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

# Strong Induction

That hypothesis where we assume $P(\text{base case}), \ldots, P(k)$ instead of just $P(k)$ is called a strong inductive hypothesis.

Strong induction is the same fundamental idea as weak (“regular”) induction.

$P(0)$ is true.
And $P(0) \rightarrow P(1)$, so $P(1)$.
And $P(1) \rightarrow P(2)$, so $P(2)$.
And $P(2) \rightarrow P(3)$, so $P(3)$.
And $P(3) \rightarrow P(4)$, so $P(4)$.
…

$P(0)$ is true.
And $P(0) \rightarrow P(1)$, so $P(1)$.
And $[\mathrm{P}(0) \wedge P(1)] \rightarrow P(2)$, so $P(2)$.
And $[\mathrm{P}(0) \wedge \cdots \wedge P(2)] \rightarrow P(3)$, so $P(3)$.
And$[\mathrm{P}(0) \wedge \cdots \wedge P(3)] \rightarrow P(4)$, so $P(4)$.
…

# Making Induction Proofs Pretty

All of our strong induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on $n$.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $\mathrm{P(b)} \land \cdots \land P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P(b)} \land \cdots \land P(k)] \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

# Strong Induction vs. Weak Induction

Think of strong induction as "my recursive call might be on LOTS of smaller values" (like mergesort – you cut your array in half)

Think of weak induction as "my recursive call is always on one step smaller."

Practical advice:

A strong hypothesis isn't wrong when you only need a weak one (but a weak one is wrong when you need a strong one). Some people just always write strong hypotheses. But it's easier to typo a strong hypothesis.

Robbie leaves a blank spot where the IH is, and fills it in after the step.

# Practical Advice

How many base cases do you need?

Always at least one.

If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.

If you always go back $s$ steps, at least $s$ consecutive base cases.

Enough to make sure every case is handled.

# Let’s Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each).
Prove that I can make exactly $n$ cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think…how would the inductive step go?

# Stamp Collection (attempt)

Define $P(n)$ "I can make $n$ cents of stamps with just $4$ and $5$ cent stamps."

We prove $P(n)$ is true for all integers $n \geq 12$ by induction on $n$.

Base Case:

12 cents can be made with three $4$ cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and] $P(k)$, for an arbitrary $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another $4$ cent stamp gives exactly $k + 1$ cents.

# Stamp Collection

Is the proof right?

How do we know $P(13)$

We're not the base case, so our inductive hypothesis assumes $P(12)$, and then we say if $P(9)$ then $P(13)$.

Wait a second....

If you go back $s$ steps every time, you need $s$ base cases.

Or else the first few values aren't proven.

# Stamp Collection

Define $P(n)$ to be "I can make $n$ cents of stamps with just $4$ and $5$ cent stamps."

We prove $P(n)$ is true for all integers $n \geq 12$ by induction on $n$.

Base Case:

12 cents can be made with three $4$ cent stamps.
13 cents can be made with two $4$ cent stamps and one $5$ cent stamp.
14 cents can be made with one $4$ cent stamp and two $5$ cent stamps.
15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose $\mathrm{P}(12) \wedge P(13) \wedge \cdots \wedge P(k)$, for an arbitrary $k \geq 15$.

Inductive Step:

We want to make $k+1$ cents of stamps. By IH we can make $k-3$ cents exactly with stamps. Adding another $4$ cent stamp gives exactly $k+1$ cents.

# A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back $s$ steps, you need $s$ base cases (otherwise $b + 1$ will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on $n$.

2. Base Cases: Show $P(b_{min}), P(b_{min+1}) \dots P(b_{max})$ i.e. show the base cases

3. Inductive Hypothesis: Suppose $P(b_{min}) \wedge P(b_{min} + 1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq b_{max}$. (The smallest value of $k$ assumes **all** bases cases, but nothing else)

4. Inductive Step: Show $P(k + 1)$ (i.e. get $[\mathrm{P}(\mathrm{b}_{min}) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$)

5. Conclude by saying $P(n)$ is true for all $n \geq b_{min}$ by the principle of induction.

# Stamp Collection, Done Wrong

Define $P(n)$ to be "I can make $n$ cents of stamps with just $4$ and $5$ cent stamps."

We prove $P(n)$ is true for all integers $n \geq 12$ by induction on $n$.

Base Case:

12 cents can be made with three $4$ cent stamps.
Inductive Hypothesis Suppose $P(k)$, $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k$ cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

$P(n)$ holds for all $n$ by the principle of induction.

# Stamp Collection, Done Wrong

What if the starting point doesn't have any 4 cent stamps?

Like, say, 15 cents = 5+5+5.

# Claim: $3|(2^{2n}-1)$ for all natural numbers

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

# Claim: $3|(2^{2n}-1)$ for all natural numbers

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all natural numbers $n$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and $0$ is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

# Claim: $3|(2^{2n}-1)$ for all natural numbers

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all natural numbers $n$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and $0$ is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer $j$ such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

FORCE the expression in your IH to appear

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

# Claim: $3|(2^{2n}-1)$ for all natural numbers

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all natural numbers $n$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and $0$ is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer $j$ such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4(2^{2k} - 1 + 1) - 1 = 4\left(2^{2k} - 1\right) + 4 - 1$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer $j$

$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

# Claim: $3|(2^{2n}-1)$ for all natural numbers

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2\cdot 0} - 1 = 0 = 3 \cdot 0$

$2^{2\cdot 1} - 1 = 3 = 3 \cdot 1$

$2^{2\cdot 2} - 1 = 15 = 3 \cdot 5$

$2^{2\cdot 3} - 1 = 63 = 3 \cdot 21$

$2^{2\cdot 4} - 1 = 255 = 3 \cdot 85$

$2^{2\cdot 5} - 1 = 1023 = 3 \cdot 341$

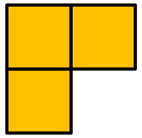
The divisor goes from $k$ to $4k + 1$

$$0 \to 4 \cdot 0 + 1 = 1$$
$$1 \to 4 \cdot 1 + 1 = 5$$
$$5 \to 4 \cdot 5 + 1 = 21$$

…

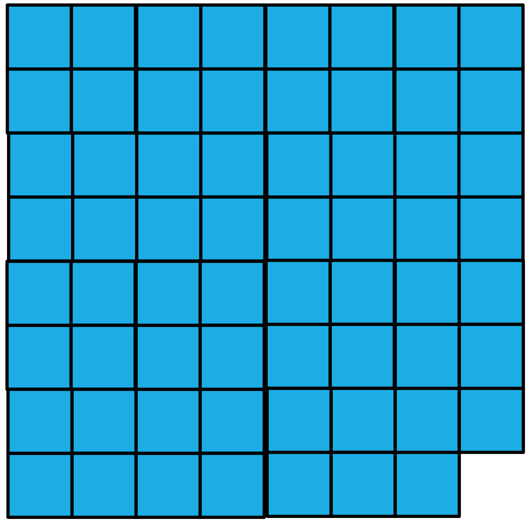
That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.

# Even more practice



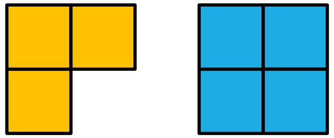
I've got a bunch of these 3 piece tiles.

I want to fill a $2^n$x$2^n$ grid ($n \geq 1$) with the pieces, except for a 1x1 spot in a corner.



# Gridding (not a full proof, just intuition)

Base Case: $n = 1$



Inductive hypothesis: Suppose you can tile a $2^k$x$2^k$ grid, except for a corner.

Inductive step: $2^{k+1}$x$2^{k+1}$, divide into quarters. By IH can tile…