Often care about 2 (or more) random variables *simultaneously*

measured X = height and Y = weight

X = cholesterol and Y = blood pressure

X_1, X_2, X_3 = work loads on servers A, B, C

Joint probability mass function:

$f_{XY}(x, y) = P(X = x \;\&\; Y = y)$

Joint cumulative distribution function:

$F_{XY}(x, y) = P(X \leq x \;\&\; Y \leq y)$

Two joint PMFs

W \ Z	1	2	3
1	2/24	2/24	2/24
2	2/24	2/24	2/24
3	2/24	2/24	2/24
4	2/24	2/24	2/24

X \ Y	1	2	3
1	4/24	1/24	1/24
2	0	3/24	3/24
3	0	4/24	2/24
4	4/24	0	2/24

P(W = Z) = 3 * 2/24 = 6/24

P(X = Y) = (4 + 3 + 2)/24 = 9/24

Can look at arbitrary relationships between variables this way

Two joint PMFs

W \ Z	*1*	*2*	*3*	$f_W(w)$
1	2/24	2/24	2/24	6/24
2	2/24	2/24	2/24	6/24
3	2/24	2/24	2/24	6/24
4	2/24	2/24	2/24	6/24
$f_Z(z)$	8/24	8/24	8/24	

X \ Y	*1*	*2*	*3*	$f_X(x)$
1	4/24	1/24	1/24	6/24
2	0	3/24	3/24	6/24
3	0	4/24	2/24	6/24
4	4/24	0	2/24	6/24
$f_Y(y)$	8/24	8/24	8/24	

Marginal distribution of one r.v.: sum over the other:

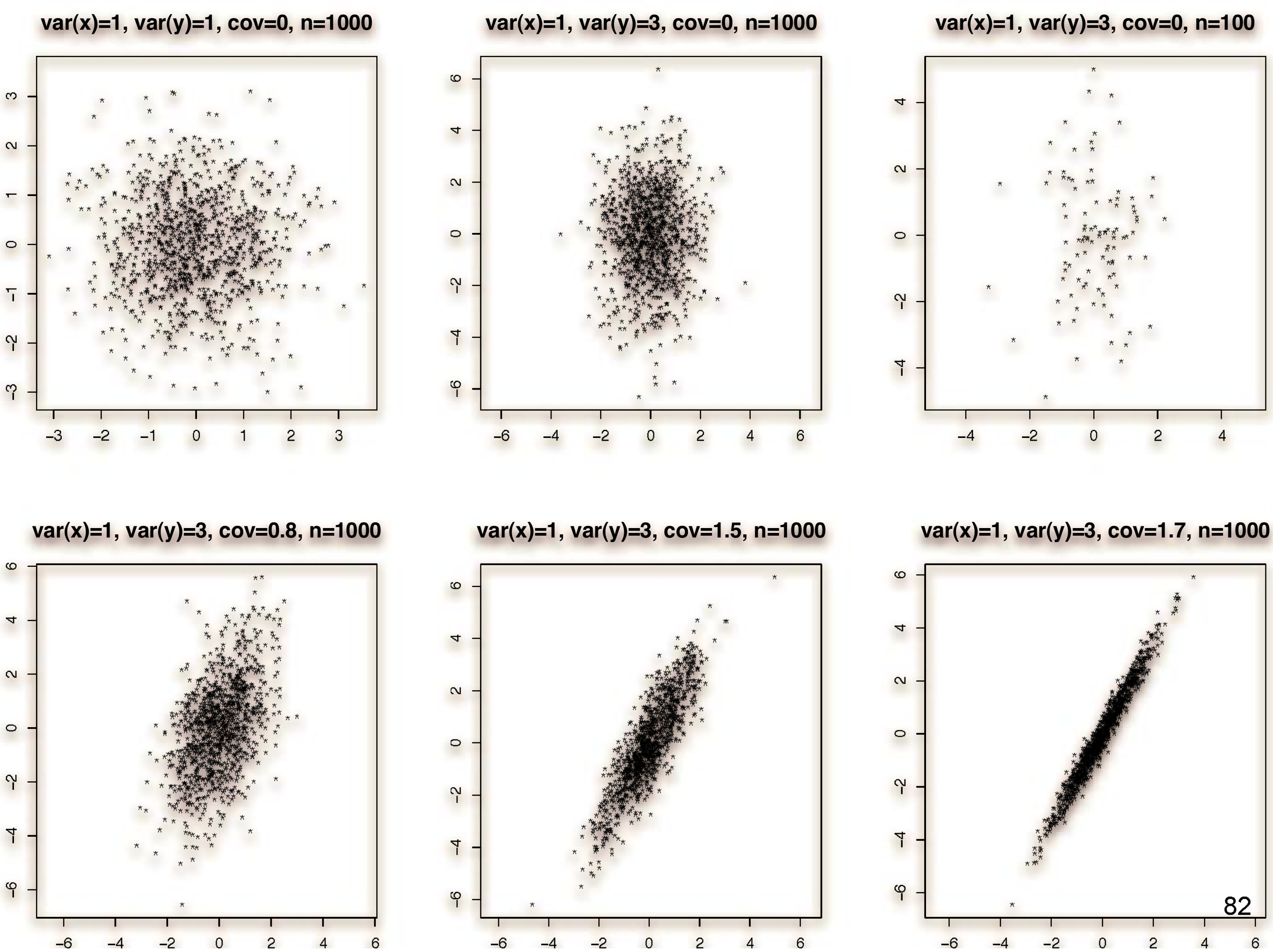
$f_Y(y) = \Sigma_x f_{XY}(x,y)$ $f_X(x) = \Sigma_y f_{XY}(x,y)$

Question: Are W & Z independent? Are X & Y independent?

sampling from a (continuous) joint distribution

Top row; independent variables

bottom row: dependent variables



A function g(X,Y) defines a new random variable.

Its expectation is:

$E[g(X, Y)] = \Sigma_x \Sigma_y\, g(x, y)\, f_{XY}(x,y)$

Expectation is linear. I.e., if g is linear:

$E[g(X, Y)] = E[a\,X + b\,Y + c] = a\,E[X] + b\,E[Y] + c$

Example:

g(X,Y) = 2X-Y

E[g(X,Y)] = 72/24 = 3

E[g(X,Y)] = 2•2.5 - 2 = 3

X \ Y	1	2	3
1	1 • 4/24	0 • 1/24	-1 • 1/24
2	3 • 0/24	2 • 3/24	1 • 3/24
3	5 • 0/24	4 • 4/24	3 • 2/24
4	7 • 4/24	6 • 0/24	5 • 2/24

RV: a numeric function of the outcome of an experiment

Probability Mass Function p(x): prob that RV = x; $\Sigma p(x)=1$

Cumulative Distribution Function F(x): *probability that RV ≤ x*

Concepts generalize to *joint* distributions

Expectation:

- of a random variable: $E[X] = \Sigma_x xp(x)$
- of a function: if $Y = g(X)$, then $E[Y] = \Sigma_x g(x)p(x)$
- linearity:
  - $E[aX + b] = aE[X] + b$
  - $E[X+Y] = E[X] + E[Y]$; even if *dependent*
  - *this interchange of "order of operations" is quite special to linear combinations. E.g. $E[XY] \neq E[X]*E[Y]$, in general (but see below)*

Variance:

$Var[X] = E[(X-E[X])^2] = E[X^2] - (E[X])^2]$

Standard deviation: $\sigma = \sqrt{Var[X]}$

$Var[aX+b] = a^2 Var[X]$

If X & Y are *independent*, then

$E[X \bullet Y] = E[X] \bullet E[Y]$;

$Var[X+Y] = Var[X]+Var[Y]$

(These two equalities hold for *indp* rv's; but not in general.)

Important Examples:

Bernoulli: $P(X=1) = p$ and $P(X=0) = 1-p$ $\mu = p$, $\sigma^2 = p(1-p)$

Binomial: $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$ $\mu = np$, $\sigma^2 = np(1-p)$

Poisson: $P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$ $\mu = \lambda$, $\sigma^2 = \lambda$

Bin(n,p) ≈ Poi(λ) where $\lambda = np$ fixed, $n \to \infty$ (and so $p=\lambda/n \to 0$)

Geometric $P(X=k) = (1-p)^{k-1}p$ $\mu = 1/p$, $\sigma^2 = (1-p)/p^2$

Many others, e.g., hypergeometric