Section 5-23

- Review of important distributions
- Another randomized algorithm

Discrete Random Variables

Bernoulli Distribution

Definition: value 1 with probability p, 0 otherwise (prob. $q = 1-p$)

Example: coin toss ($p = ½$ for fair coin)

Parameters: p

Properties:

$E[X] = p$

$Var[X] = p(1-p) = pq$

Binomial Distribution

Definition: sum of n independent Bernoulli trials, each with parameter p

Example: number of heads in 10 independent coin tosses

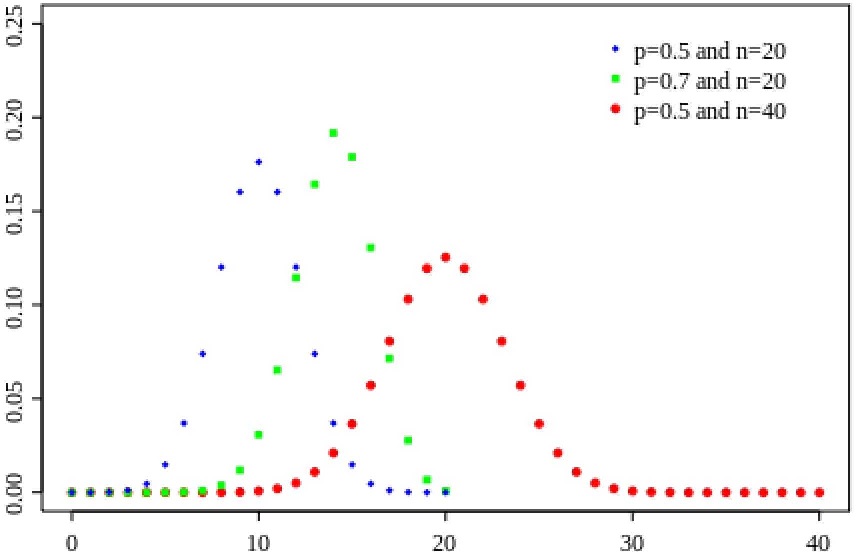
Parameters: n, p

Properties:

E[X] = np

Var(X) = $np(1-p)$

pmf: $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$



Poisson Distribution

Definition: number of events that occur in a unit of time, if those events occur independently at an average rate λ per unit time

Example: # of cars at traffic light in 1 minute, # of deaths in 1 year by horse kick in Prussian cavalry

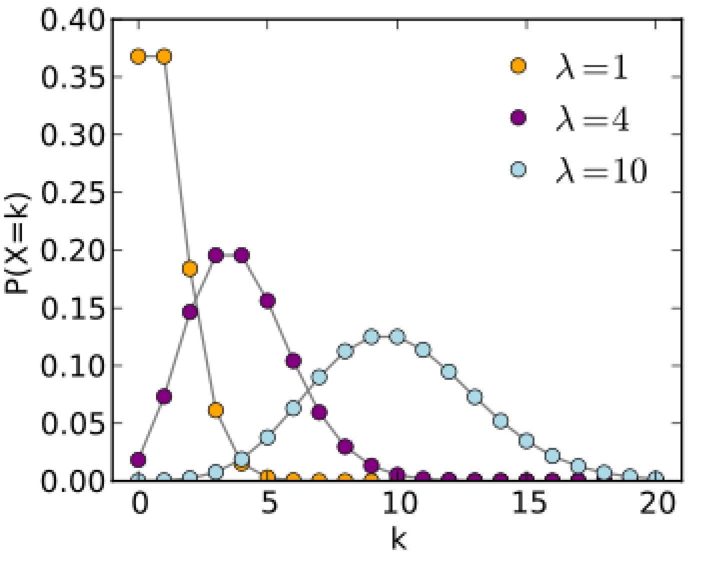
Parameters: λ

Properties:

E[X] = λ

Var[X] = λ

pmf: $Pr(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$



Geometric Distribution

Definition: number of independent Bernoulli trials with parameter p until and including first success (so X can take values 1, 2, 3, ...)

Example: # of coins flipped until first head

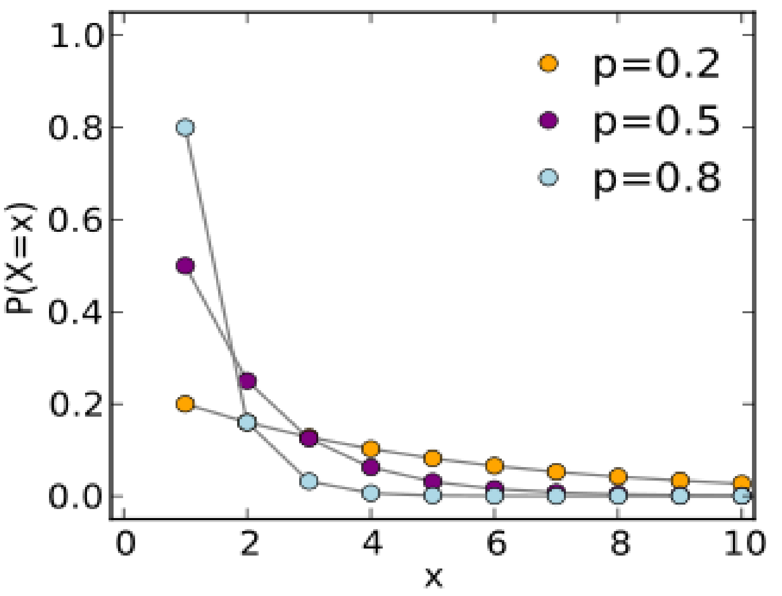
Parameters: p

Properties:

E[X] = $\frac{1}{p}$

Var[X] = $\frac{1-p}{p^2}$

pmf: $\Pr(X = k) = (1-p)^{k-1}p$



Hypergeometric Distribution

Definition: number of successes in *n* draws (without replacement) from *N* items that contain *K* successes in total

Example: An urn has 10 red balls and 10 blue balls. What is the probability of drawing 2 red balls in 4 draws?

Parameters: *n*, *N*, *K*

Properties:

E[X] = $n\frac{K}{N}$

Var[X] = $n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$

pmf: $Pr(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

Think about the pmf; we've been doing it for weeks now: ways-to-choose-successes times ways-to-choose-failures over ways-to-choose-n

Also, consider that the binomial dist. is the with-replacement analog of this

Continuous Random Variables

Uniform Distribution

Definition: A random variable that takes any real value in an interval with equal likelihood

Example: Choose a real number (with infinite precision) between 0 and 10

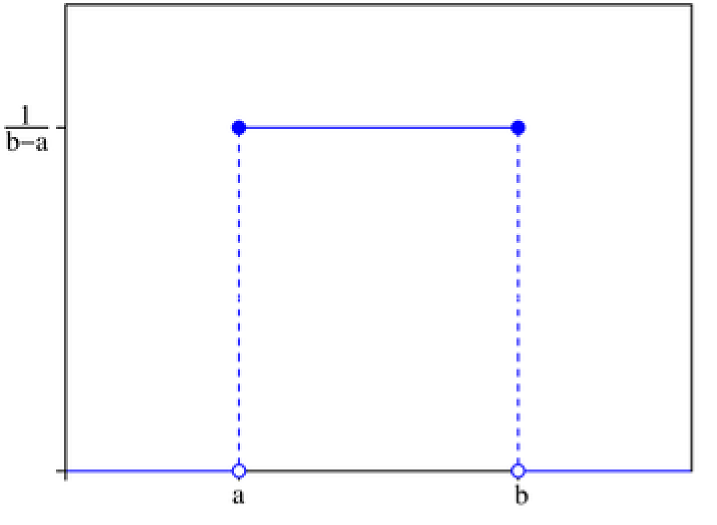
Parameters: *a*, *b* (lower and upper bound of interval)

Properties:

E[X] = $\frac{a+b}{2}$

Var[X] = $\frac{(b-a)^2}{12}$

pdf: $f(x) = \frac{1}{b-a}$ if $x \in [a, b]$, 0 otherwise



Exponential Distribution

Definition: Time until next events in Poisson process

Example: How long until the next soldier is killed by horse kick?

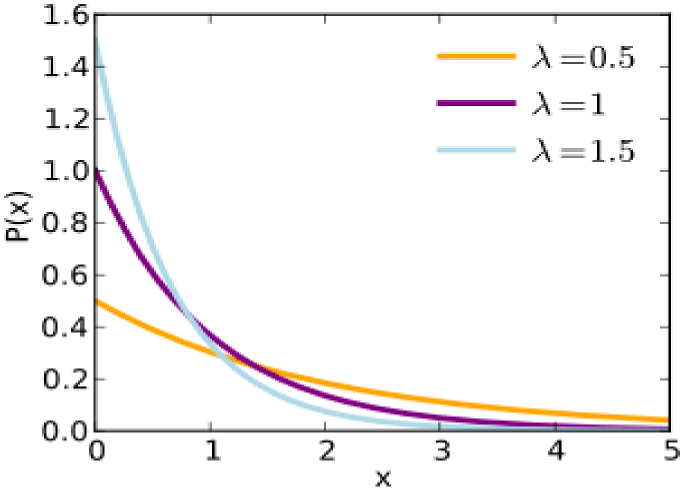
Parameters: λ, the rate at which Poisson events occur

Properties:

$E[X] = \frac{1}{\lambda}$

$Var[X] = \frac{1}{\lambda^2}$

pdf: $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, 0 for $x < 0$



Normal Distribution

Definition: Your classic bell curve

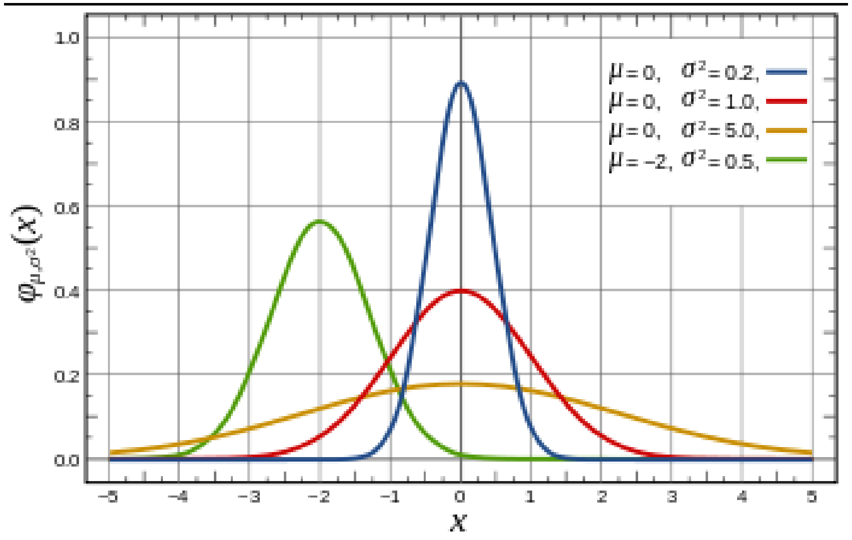
Example: Quantum harmonic oscillator ground state (exact)
Human heights, binomial random variables (approx)

Properties: μ, σ^2 (yes, mean and variance are given)

$E[X] = \mu$

$Var[X] = \sigma^2$

pdf: $f(x) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$



Another Randomized Algorithm

Matrix Multiplication

Multiplying $n \times n$ matrices (n = 2 in this example)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Complexity of straightforward algorithm: O(n^3) time

(There are 8 multiplications here; in general, n multiplications for each of n^2 entries)

Coppersmith–Winograd algorithm (with help by others) can perform this operation in time $O(n^{2.38})$

(2.3755 in 1990, 2.3727 by 2011. Progress!)

Frievalds' Algorithm

- Determine whether $n \times n$ matrices A, B and C satisfy the condition $AB = C$
- Method:
  - Choose $x \in \{0,1\}^n$ randomly and uniformly (vector of length n)
  - If $ABx \neq Cx$, then $AB \neq C$
  - Else, $AB = C$ *probably*

Results of Frievalds' Algorithm

- Runs in time $O(n^2)$
  - $ABx = A(Bx)$, so we have 3 instances of an $n \times n$ matrix times an n-vector
  - these are $O(n^2)$ time operations
- Via some math magic,

$$P(\text{the algorithm reports } AB = C \mid AB \neq C) \leq ½$$

- By iterating k random choices of x, can decrease probability of error to $1/2^k$.
- Interesting comparison
  - Quicksort is always correct, runs slowly with small probability
  - Frievalds' alg. is always fast, incorrect with small probability