Another Randomized Algorithm

Freivalds' Algorithm

for

Matrix Multiplication

Randomized Algorithms

- Quicksort makes effective use of random numbers, but is no faster than Mergesort or Heapsort.
- Here we will see a problem that has a simple randomized algorithm faster than any known deterministic solution.

Matrix Multiplication

Multiplying $n \times n$ matrices ($n = 2$ in this example)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Complexity of straightforward algorithm: $\Theta(n^3)$ time

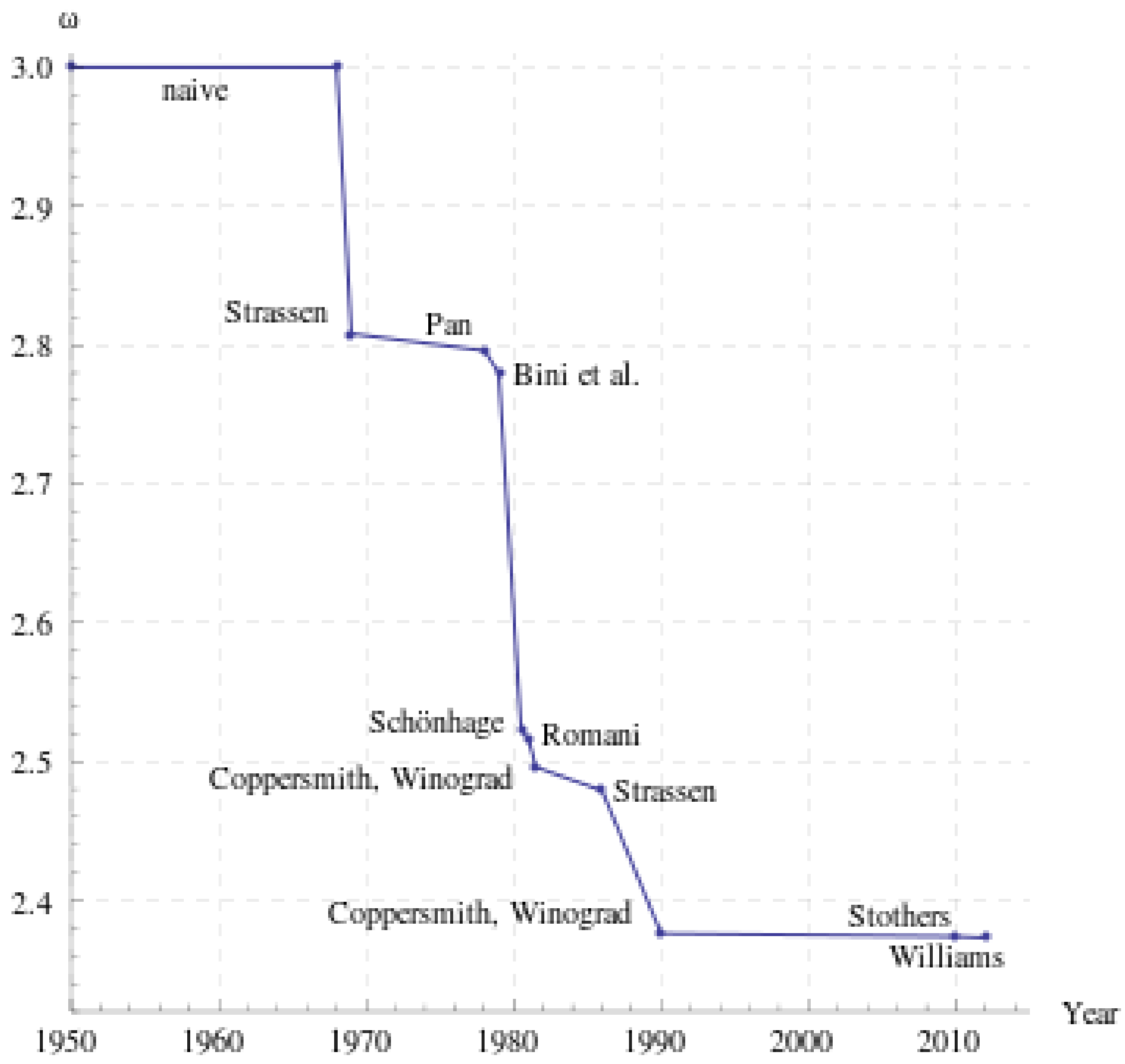
(There are 8 multiplications here; in general, n multiplications for each of n^2 entries)

Coppersmith & Winograd showed how to do it in time $O(n^{2.376})$ in 1989.

Williams improved this to $O(n^{2.3729})$ in 2011. Progress!

History of Matrix Multiplication Algorithms

Running time: $O(n^{\omega})$



Frievalds' Algorithm (1977)

- Freivalds' variant of problem:

  Determine whether $n \times n$ matrices A, B, and C satisfy the condition $AB = C$

- Method:
  - Choose $x \in \{0,1\}^n$ randomly and uniformly (vector of length n)
  - If $ABx \neq Cx$ then report "$AB \neq C$"

    else report "$AB = C$ *probably*"

Running Time

- $ABx = A(Bx)$, so we have 3 instances of an $n \times n$ matrix times an n-vector
- These are $O(n^2)$ time operations if done straightforwardly
- Total running time $O(n^2)$
- Fastest deterministic solution known: $O(n^{2.3729})$

How Often Is It Wrong?

$P(ABx = Cx \mid AB = C) = 1$

$P(ABx = Cx \mid AB \neq C) \leq ½$:

- Assume $AB \neq C$
- Then $AB - C \neq 0$, so there exist i, j with $(AB - C)_{ij} \neq 0$
- Let $(d_1, d_2, \ldots, d_n)$ be i-th row of $AB - C$; $d_j \neq 0$
- $P((AB - C)\, x = 0 \mid AB \neq C)$

$$\leq P\left(\sum_{i=1}^{n} d_i x_i = 0 \mid AB \neq C\right)$$

$$= P\left(x_j = -\frac{1}{d_j} \sum_{i \neq j} d_i x_i \mid AB \neq C\right)$$

$$\leq ½$$

Decreasing the Probability of Error

- By iterating with k random, independent choices of x, we can decrease probability of error to $1/2^k$, using time $O(kn^2)$.
- Interesting comparison
  - Quicksort is always correct, and runs slowly with small probability.
  - Frievalds' algorithm is always fast, and incorrect with small probability.