CSE 312: Foundations of Computing II
Quiz Section #6: Exponential distribution

Recall the probability density function for $X \sim \text{Exp}(\lambda)$:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}.$$

1. Prove that $\text{E}[X] = 1/\lambda$. (Hint: use integration by parts.)
2. Prove that $\text{P}(X \geq t) = e^{-\lambda t}$, for $t \geq 0$. As a corollary, show that the cumulative distribution function for X is $F(t) = 1 - e^{-\lambda t}$.
3. Prove the memorylessness property for exponential distributions: If s and t are nonnegative, then $\text{P}(X \geq s + t \mid X \geq s) = \text{P}(X \geq t)$.
4. In Schnapsen, it usually feels as though you have to wait forever before you are finally dealt the powerful trump marriage. Let's explore this.
   (a) What is the probability p of being dealt the trump marriage? Don't forget that part of this event is that neither a king nor a queen is dealt as the face-up trump card.
   (b) Let X be the number of deals up to and including the first time that you are dealt the trump marriage. What is the name of the distribution that describes X? What is $\text{E}[X]$; that is, how many deals do you expect to wait until you are dealt the trump marriage? (You don't have to derive the formula for $\text{E}[X]$; it is a formula you should already know.)
   (c) Write the formula for $\text{P}(X > t)$. What is the probability that you go $t = 35$ deals without ever being dealt the trump marriage? Calculate your answer to 3 significant digits.
   (d) It seems reasonable that the exponential distribution would be a fairly good approximation to this distribution, since you would expect that the probability of never having been dealt the trump marriage decays exponentially with the number of deals played. Let $Y \sim \text{Exp}(p)$. What is $\text{E}[Y]$? (You don't have to derive the formula for $\text{E}[Y]$; it is a formula you should already know.) Compare your formula to your answer for part (b).
   (e) Write the formula for $\text{P}(Y > t)$. What is $\text{P}(Y > 35)$? Calculate your answer to 3 significant digits. Compare your answer with your answer for part (c).
   (f) Going back to part (c), if $X > 35$ then $X \geq 36$, because X is the number of deals, which must be an integer. There is no such constraint on Y, since the exponential is a continuous distribution. Perhaps, then, we would get a better approximation by calculating $\text{P}(Y > 35.5)$. (This is called the "continuity correction", and we will see it again soon.) Calculate your answer to 3 significant digits. Compare your answer with your answers for parts (c) and (e).