Continuous Random Variables (BT Chapter 3)

A. A continuous r.v. takes values from an uncountable set.

Ex: Weight of a randomly chosen person

Waiting time until arrival of next packet.

B. *Defn*: $f: \mathbb{R} \to \mathbb{R}$ is a *probability density function* (or density) iff $\forall x\ f(x) \geq 0$, and $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. (Normalized, like a discrete PMF).

B. *Defn*: The *cumulative distribution function* $F: \mathbb{R} \to \mathbb{R}$ associated with pdf f is

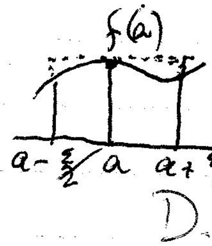
$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx.$$

Note: $P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)\,dx$

Note: $f(x) = \frac{d}{dx} F(x)$.

C. Remarks:

1. Densities are not probabilities; they may be > 1.
2. $P(X = a) = F(a) - F(a) = 0$



But $P(a - \frac{\varepsilon}{2} < X \leq a + \frac{\varepsilon}{2}) = F(a + \frac{\varepsilon}{2}) - F(a - \frac{\varepsilon}{2}) \approx \varepsilon f(a)$,

so prob X is near a is proportional to density $f(a)$.

D. For continuous r.v. X, usually substitute $\int$ for Σ.

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx.$$

$$E[aX + b] = a E[X] + b$$

$$E[X + Y] = E[X] + E[Y]$$

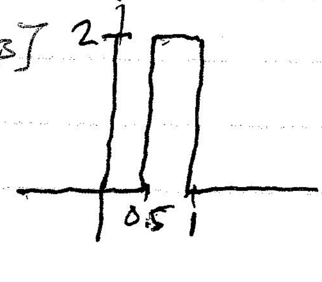
$$E[g(X)] = \int g(x) f(x)\,dx$$

$$Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

E. Uniform:

$X \sim Uni(\alpha, \beta)$: $f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } x \in [\alpha, \beta] \\ 0, & \text{otherwise} \end{cases}$

Graph it for $Uni(0.5, 1)$.



$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\,dx$$
$$= \frac{1}{\beta-\alpha}\cdot\frac{1}{2}x^2\Big|_{\alpha}^{\beta} = \frac{1}{2}\,\frac{\beta^2-\alpha^2}{\beta-\alpha} = \frac{1}{2}(\alpha+\beta).$$

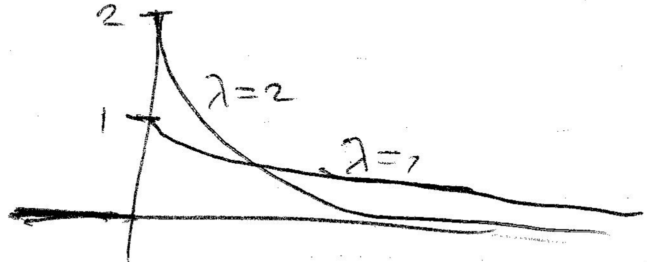
F. Exponential: waiting for next event, where events happen at rate λ per unit time.

Radioactive decay: ~~how long~~ time to next α particle.

Customers: time till next packet arrives at server.

$X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}$$



$E[X] = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$

$P(X \geq t) = e^{-\lambda t} = 1 - F(t)$

Memorylessness: $P(X > s+t \mid X > s) = P(X > t)$, for $s, t > 0$

Poisson: How many events in a given time unit?

Exponential: Time until next event. Geometric: discrete analog

G. Normal distribution: 07continuous, slides 20–33

10 limits, slides 25–38