Tail Bounds

A. Often helpful to be able to bound the probability of being far from the mean.
Ex: Expected business cost is $1500/day. What is prob that it is > $6000 in a particular day?
Draw picture of bell shape and indicate tails

B. Markov's inequality

Theorem: If X is a nonnegative r.v., then for any $\alpha > 0$, $P(X \geq \alpha) \leq E[X]/\alpha$.
Equivalently, $P(X \geq k E[X]) \leq 1/k$.

Ex: If X is daily business cost and $E[X] = 1500$,
$P(X \geq 6000) \leq \frac{1500}{6000} = 1/4$.

Proof: $E[X] = \sum_x x p(x) = \sum_{x<\alpha} x p(x) + \sum_{x \geq \alpha} x p(x)$
$\geq 0 + \sum_{x \geq \alpha} \alpha p(x) = \alpha P(X \geq \alpha)$

C. Chebyshev's inequality: Suppose we also know $Var(X)$.

Theorem: If Y is a r.v. with $E[Y] = \mu$, then for any $\alpha > 0$, $P(|Y - \mu| \geq \alpha) \leq Var(Y)/\alpha^2$.

Proof: Let $X = (Y-\mu)^2$. X is nonnegative, so
$P(|Y-\mu| \geq \alpha) = P(X \geq \alpha^2) \leq E[X]/\alpha^2 = Var(Y)/\alpha^2$.

Ex: If Y is daily business cost, $E[Y] = 1500$, ~~$Var(Y) = 250{,}000$~~, $\sigma = 500$, $P(Y \geq 6000) \leq P(|Y-\mu| \geq 4500) \leq \left(\frac{500}{4500}\right)^2 = 1/81$.

Corollary: Let $\mu = E[Y]$ and $\sigma = \sqrt{Var(Y)}$. Then for any $t > 0$,
$P(|Y-\mu| \geq t\sigma) \leq 1/t^2$.

D. Chernoff bounds.

Theorem: Suppose $X \sim Bin(n, p)$ and $\mu = E[X]$. For any $0 < \delta < 1$,
$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2 \mu}$ and
$P(X \leq (1-\delta)\mu) \leq e^{-\frac{1}{2}\delta^2 \mu}$.

(Note: $Bin(n,p)$ not symmetric when $p \neq \frac{1}{2}$.)