CSE 312 Practice Midterm #1 Solutions

Q1. A small scale distributed file system consists of 12 servers. We number them from 1 to 12 respectively. According to historical data, each server crashes independently with probability $\frac{1}{6}$ every hour. Answer the following for one particular hour:

a) What is the probability both the 3rd server and the 6th server crash?

Since the events of each server crashing are independent,
$$P(3rd \text{ and } 6th \text{ servers crash})$$
$$= P(3rd \text{ server crashes}) \times P(6th \text{ server crashes})$$
$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

b) The entire system will continue to function as long as more than (but not including) $\frac{3}{4}$ of the servers are working. What is the probability that the entire system will fail? (No need to simplify your answer.)

Define a r.v. X to be the number of servers working.
$$X \sim Bin(12, 5/6)$$
$$P(x \le 9) = 1 - (P(x = 12) + P(x = 11) + P(x = 10))$$
$$= 1 - \left[\binom{12}{12}\left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{12} + \binom{12}{11}\left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{11} + \binom{12}{10}\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10} \right]$$

Q2. There's a popular social network where two people are considered to be connected if both of them add each other as "Friends". The structure of this social network is determined by which pairs of people are "Friends". If there are n active users on the social network, how many possibilities are there for the structure of this social network?

There are $\binom{n}{2}$ possible ways that we can pick two people, i. e., one possible friendship. For each possible friendship, it may or may not exist, so there are $2^{\binom{n}{2}}$ possible structures for the social network.

Q3. I have a pile of 6 (identical-looking) coins:
- 3 of them are fair coins: P(heads) = ½
- 2 of them are biased such that P(heads) = 1
- 1 of them is biased such that P(heads) = ¾

Suppose I draw a coin randomly, with each coin being equally likely. Then I flip the coin 6 times, and it comes up heads 4 times. What is the probability that the coin I flipped was one of the fair coins? No need to simplify your answer.

Let $C1$ be the event that I selected a fair coin, $C2$ be the event that I selected a coin with $P(heads) = 1$, and $C3$ be the event that I selected a coin with $P(heads) = $ ¾. Let A be the event that there are exactly 4 heads out of the 6 flips. The prior probabilities are

$$P(C1) = \frac{3}{6}, P(C2) = \frac{2}{6}, P(C3) = \frac{1}{6}$$

Then

$$P(A|C1) = \binom{6}{4} \times 0.5^4 \times 0.5^2$$

$$P(A|C2) = 0$$

This is because all flips are guaranteed to be heads in the event $C2$, so it's impossible to get only 4 out of 6 heads.

$$P(A|C3) = \binom{6}{4} \times 0.75^4 \times 0.25^2$$

Using Bayes' Theorem,

$$P(C1|A) = \frac{P(A|C1)P(C1)}{P(A|C1)P(C1) + P(A|C2)P(C2) + P(A|C3)P(C3)}$$

We can just plug our values into this and get the result.

Q4. There is a train with 3 carriages which are all initially empty. It arrives at a train station where 20 people are waiting to get on. Each person chooses to get on each of the carriages with equal probability. How many ways are there for people to get on the train so that none of the carriages are empty? You may leave your answer as a simplified expression.

Let A be the number of ways for people to get on the train so that none of the carriages are empty. We can solve this problem by taking the complement and using inclusion-exclusion. Let C_i be the event that carriage i is empty.

$$A = [Total\ \#\ of\ ways\ for\ people\ to\ get\ into\ carriages] - [\geq 1\ carriage\ empty]$$
$$= [total] - |C1 \cup C2 \cup C3|$$
$$= [total] - (|C1| + |C2| + |C3|) + (|C1 \cap C2| + |C2 \cap C3| + |C1 \cap C3|)$$
$$= [total] - (singles) + (doubles)$$
$$= 3^{20} - \binom{3}{1}(3-1)^{20} + \binom{3}{2}(3-2)^{20}$$
$$= 3^{20} - 3 \times 2^{20} + 3$$

Q5. Suppose we throw n balls into n bins with the probability of a ball landing in each of the n bins being equal. What is the expected number of empty bins?

Solution. Let X be the random variable denoting the number of empty bins. Let X_i be a random variable that is 1 if the ith bin is empty and is 0 otherwise. Clearly

$$X = \sum_{i=1}^{n} X_i$$

By linearity of expectation, we have

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{i=1}^{n} \mathbf{E}[X_i] \\
&= \sum_{i=1}^{n} \Pr[X_i = 1] \\
&= \sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^{n} \\
&= \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{n} = n\left(1 - \frac{1}{n}\right)^{n}
\end{aligned}
$$

As $n \to \infty$, $(1 - \frac{1}{n})^{n} \to \frac{1}{e}$. Hence, for large enough values of n we have

$$\mathbf{E}[X] = \frac{n}{e}$$

Q6. A negative binomial r.v. is denoted $X \sim NegBin(r, p)$. X is the number of independent trials up to and including the rth success, where $P(success) = p$ for any single trial.

a) X can be written as a sum of iid (independent and identically distributed) r.v.'s, $X_1 + X_2 + \ldots + X_r$, where each X_i follows the Geometric distribution with parameter p.

b) Find the pmf $p(x)$ for X. For which values of x is this pmf defined?
$$\binom{x-1}{r-1}(1-p)^{x-r}p^{r}$$
$$x = r, r+1, \ldots$$

c) Find $E[X]$.
By Linearity of Expectation,
$$E[X] = E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = r/p$$

d) Find $Var(X)$.
By Linearity of Variance (since the trials are independent),
$$Var(X) = Var\left(\sum_{i=1}^{r} X_i\right) = \sum_{i=1}^{r} Var(X_i) = \frac{r(1-p)}{p^2}$$