CSE 312 Practice Midterm #2 Solutions

1. True/False, Short Answer. Provide a short justification for your answer.

a) True or False. For any random variables X and Y, $Var(aX + bY + c) = a^2Var(X) + b^2Var(Y)$.

False, only if they are independent.

b) Let X be a discrete random variable with pmf $p(x) = a^x, x = 1,2, \ldots$. Find a, or say if there is not enough information.

The sum must be 1, so $a = \frac{1}{2}$.

c) True or False. Suppose $F_1, F_2, \ldots, F_n$ partition the sample space, and let E be any event. Then,

$$P(E) = P(E|F_1) + P(E|F_2) + \cdots + P(E|F_n)$$

False. $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_n)P(F_n)$.

d) True or False. For any random variable X, $\log E[X] = E[\log X]$.

False. In general, $E[f(X)] \neq f(E[X])$.

e) If $Var(2X) = E[3X^2] = 12$, find $E[X]$, or say if there is not enough information.

$4Var(X) = 12 \rightarrow Var(X) = 3$. $E[3X^2] = 12 \rightarrow E[X^2] = 4$.
$Var(X) = E[X^2] - E^2[X]$, so $E[X] = \pm\sqrt{4-3} = \pm 1$.
Not enough information.

f) True or False. If E and F are independent with $P(E) \neq 0, P(F) \neq 0$, they are mutually exclusive.

False. The probability of their intersection would be the product of their probabilities, which must be 0 if they are mutually exclusive.

g) True or False. If E and F are mutually exclusive with $P(E) \neq 0, P(F) \neq 0$, they are independent.

False. The probability of their intersection would be 0, but if the probabilities of both are positive, the product of their probabilities is nonzero.

h) True or False. If $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \cup B) = 0.58$, then A and B are independent.

True. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12 = P(A)P(B)$.

2. Annie, Bill, Charlie, Danielle, Ethan, Felicia, and George are sitting in a row of nine seats (Note: there are only seven people). Annie and Bill are exes, so they cannot sit next to each other. Charlie and Danielle are dating, so they must sit next to each other. Ethan, Felicia, and George are best friends, so they also want to sit next to each other, but Felicia must be in the middle of Ethan and George (with no spaces between the three). In how many ways can they sit in the row of nine seats? (Hint: Start by grouping Ethan, Felicia, and George, as well as Charlie and Danielle. Then work on placing Annie and Bill).

Step 1: _ _ _ _ _ _ _ _ _, but count EFG as 1, CD as 1

Step 2: CD EFG _ _ _ _

2 cases

case 1: A sits on edge seat, then B has 4 options to sit × 2 edges

case 2: A sits in middle 4 seats, then B has 3 options to sit × 4 middle seats

case 1 + case 2 = 8 + 12 = 20

multiply by the following: 4×3 for CD, EFG, & 2 empty seats to sit

2! for C and D to sit

2! for E, F, and G to sit

20 × 12 × 2 × 2 = 960

3. The hypergeometric distribution represents the following scenario: Suppose there are N items in a bag, with K of them marked as successes in total (and the rest are marked as failures). We draw n of them, without replacement. Each item is equally likely to be drawn. Then, if X is the random variable that denotes the number of successes we get in the n draws, we say $X \sim HypGeo(N, K, n)$.

a) Based on the situation, specify restrictions on the relationships among $N, K, and\ n$.

$$\mathbf{0 \leq n \leq N \text{ } and \text{ } 0 \leq K \leq N}$$

b) Find the probability mass function for the hypergeometric random variable X, $p(k)$. Be sure to specify the domain for the pmf.

$$\boldsymbol{p(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

$$\mathbf{for\ 0 \leq k \leq K,}$$

$$\mathbf{0 \leq n - k \leq N - K}$$

Explanation: There are N items, and K successes, so there are $N - K$ failures. We choose k successes out of K possible. Then, we choose $n - k$ failures out of $N - K$ total. There are a total of n items drawn out of N total.

c) If we had the same setup but performed this process with replacement, then our distribution would change and the random variable Y representing the number of successes would be $Y \sim$_____(________).

$$\boldsymbol{Bin\left(n, \frac{K}{N}\right)}$$

Explanation: The probability of success each time would be $p = \frac{K}{N}$.

4. Patients arriving at a hospital emergency room are categorized as either being in critical, serious, or stable condition. In the last year, we know

- 60% of the patients were stable
- 10% of the patients were critical
- 40% of critical patients died
- 90% of serious patients survived
- 1% of stable patients died

Given that a patient survived, what is the probability that the patient was categorized as serious?

Solution. Let $F_1 = \{\text{patient is critical}\}$, $F_2 = \{\text{patient is serious}\}$, $F_3 = \{\text{patient is stable}\}$, $A = \{\text{patient survived}\}$. Then

$$\mathbf{P}(F_1) = 0.1, \quad \mathbf{P}(F_2) = 0.3, \quad \mathbf{P}(F_3) = 0.6,$$

$$\mathbf{P}(A \mid F_1) = 0.6, \quad \mathbf{P}(A \mid F_2) = 0.9, \quad \mathbf{P}(A \mid F_3) = 0.99.$$

Thus, by the second Bayes formula

$$\mathbf{P}(F_2 \mid A) = \frac{\mathbf{P}(F_2)\mathbf{P}(A \mid F_2)}{\mathbf{P}(F_1)\mathbf{P}(A \mid F_1) + \mathbf{P}(F_2)\mathbf{P}(A \mid F_2) + \mathbf{P}(F_3)\mathbf{P}(A \mid F_3)} = \frac{0.9 \cdot 0.3}{0.99 \cdot 0.6 + 0.9 \cdot 0.3 + 0.6 \cdot 0.1}$$

$$= \boxed{29\%}$$

5. Let $X_1, X_2, \dots, X_n$ be iid (independent and identically distributed) random variables each with pmf

$$p(x) = \begin{cases} 1/3, & x \in \{2,3\} \\ p, & x \in \{0,1\} \\ 0, & otherwise \end{cases}$$

Furthermore, let $X = X_1 + X_2 + \cdots + X_n$

a) Find p.

$p = \frac{1}{6}$ since probabilities have to sum to 1.

b) Find $E[-3X - 4]$.

$$E[X_i] = \frac{1}{3} * 2 + \frac{1}{3} * 3 + \frac{1}{6} * 1 + \frac{1}{6} * 0 = \frac{2}{3} + 1 + \frac{1}{6} = \frac{11}{6}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \frac{11}{6} n$$

$$E[-3X - 4] = -3E[X] - 4 = -\frac{11}{2} n - 4$$

c) Find $Var(-3X - 4)$, and $Var(3X + 4)$.

$$Var(-3X - 4) = (-3)^2 Var(X) = 9Var(X) = Var(3X + 4)$$

$$E[X_i^2] = \frac{1}{3} * 2^2 + \frac{1}{3} * 3^2 + \frac{1}{6} * 1^2 + \frac{1}{6} * 0^2 = \frac{4}{3} + 3 + \frac{1}{6} = \frac{9}{2}$$

$$Var(X_i) = E[X_i^2] - E^2[X_i] = \frac{9}{2} - \left(\frac{11}{6}\right)^2 = \frac{9}{2} - \frac{121}{36} = \frac{162 - 121}{36} = \frac{41}{36}$$

Since $X_1, \dots, X_n$ are independent,

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = \frac{41}{36} n$$

$$Var(-3X - 4) = Var(3X + 4) = 9Var(X) = \frac{41}{4} n$$