CSE 312: Foundations of Computing II
Quiz Section #7: Continuous random variables

Recall the probability density function for $X \sim \text{Exp}(\lambda)$:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}.$$

1. Starting from the probability density function, prove that $\text{E}[X] = 1/\lambda$. (Hint: use integration by parts.)

2. Starting from the probability density function, prove that $\text{P}(X \geq t) = e^{-\lambda t}$, for $t \geq 0$. As a corollary, show that the cumulative distribution function for X is $F(t) = 1 - e^{-\lambda t}$.

3. Prove the memorylessness property for the exponential distribution $\text{Exp}(\lambda)$: If s and t are nonnegative, then $\text{P}(X > s + t \mid X > s) = \text{P}(X > t)$.

4. Prove the memorylessness property for the geometric distribution $\text{geo}(p)$.

5. The *gamma distribution* $\text{Gamma}(r, \lambda)$ is defined to be the sum of r independent $\text{Exp}(\lambda)$ random variables. It represents the waiting time until the r-th event. If $X \sim \text{Gamma}(r, \lambda)$, find $\text{E}[X]$ and $\text{Var}(X)$. (Just as the exponential is the continuous analog of the geometric – both representing waiting time – the gamma is the continuous analog of the negative binomial – see Exercise #6 of practice midterm #1.)

6. Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f(x) = \frac{c}{1+x^2}$ is valid. Then find $\text{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

7. Let $X \sim \text{Exp}(\lambda)$. For $t < \lambda$, find $M_X(t) = \text{E}[e^{tX}]$. M is called the *moment generating function* of X. Find $M'_X(0)$ and $M''_X(0)$. Do you notice any relationship between these two values and $\text{E}[X]$ and $\text{E}[X^2]$ (which are sometimes called the first and second *moments* of X)?

8. Let $X \sim \text{N}(50, 5)$. What is the probability that X is greater than 45 and less than 52? The Φ table is on the next page.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 1: Table for the cumulative distribution function of standard normal N(0, 1)