Solutions to
Midterm Practice Problems II

1. Answer the following questions with True or False:

   (a) $(p \to q) \leftrightarrow (\neg q \to \neg p)$ T

   (b) $(p \to q) \leftrightarrow (\neg p \vee q)$ T

   (c) If A and B are sets, $A - B = \overline{A \cap B}$. F

   (d) If $a|b$ and $a|c$, then $b|c$. F

   (e) If $a|(b + c)$, then $a|b$ and $a|c$. F

   Let $P(x, y)$ be the statement "$x \geq y$". Let the universe of x and y be the natural numbers (all integers greater than or equal to zero).

   (f) $\forall x \exists y P(x, y)$ T

   (g) $\exists x \forall y P(x, y)$ F

   (h) $\exists y \forall x P(x, y)$ T

   (i) $\neg \exists x \forall y \neg P(x, y)$ T

   The function $y = x^2$ is:

   (j) one-to-one on the set of integers. F

   (k) onto on the set of integers. F

   (l) one-to-one on the set of **positive** integers. T

2. Show using set builder notation that for sets A and B,

$$(A \cup (B - A) = A \cup B).$$

$$\begin{aligned}
&A \cup (B - A) = && \\
&= \{x | (x \in A) \vee (x \in B \wedge x \notin A)\} && \text{def. of union \& set difference} \\
&= \{x | (x \in A) \vee (x \in B \wedge x \in \bar{A})\} && \text{def. of complement} \\
&= \{x | (x \in A \vee x \in B) \wedge (x \in A \vee x \in \bar{A})\} && \text{distributive} \\
&= \{x | (x \in A \vee x \in B) \wedge T\} && \text{tautology} \\
&= \{x | x \in A \vee x \in B\} && \text{identity} \\
&= A \cup B && \text{def. of union}
\end{aligned}$$

3. Show that $4n + 3$ and $5n + 4$ are relatively prime. (Hint: Use Euclid's algorithm. Do not use induction.)

   We show that 1 is $\gcd(4n + 3, 5n + 4)$.

$$\begin{aligned}
5n + 4 &= (4n + 3)1 + (n + 1) && (1) \\
4n + 3 &= (n + 1)3 + n && (2) \\
n + 1 &= n1 + 1 && (3) \\
n &= 1n + 0 && (4)
\end{aligned}$$

4. Prove using induction that $n^2 - 7n + 12$ is nonnegative for $n \geq 3$.

   Base Case: $n = 3$: $9 - 21 + 12 = 0$. 0 is nonnegative, so true.

   Inductive step:
   Assume that $n^2 - 7n + 12$ is nonnegative for up to n.

   Consider $n + 1$:

$$\begin{aligned}
(n + 1)^2 - 7(n + 1) + 12 &= n^2 + 2n + 1 - 7n - 7 + 12 \\
&= (n^2 - 7n + 12) + (2n + 1 - 7)
\end{aligned}$$

   By our inductive step, we know that $n^2 - 7n + 12 \geq 0$. We only need to show that $2n - 6 \geq 0$. Since this a line with positive slope and since $n \geq 3$, we know that $2n - 6 \geq 0$.

   Since we have proven both parts to be nonnegative, then

$$(n + 1)^2 - 7(n + 1) + 12 \geq 0$$

5. Construct a logical argument using rules of inference to show that the following sentences imply the conclusion "It rained:"

   "If it does not rain or if it is not foggy, then the sailing race will be held and the life-saving demonstration will go on."

   "If the sailing race is held, then the trophy will be awarded."

   "The trophy was not awarded."

   Justify each step by indicating the rule you applied.

   Define the following:

   - $R =$ "It rains"
   - $F =$ "It is foggy"
   - $S =$ "The sailing race will be held"
   - $D =$ "Life-saving demonstrations will go on"
   - $T =$ "The trophy will be awarded"

   We can now proceed to prove the claim:

(1)	$\neg T$	hypothesis
(2)	$S \rightarrow T$	hypothesis
(3)	$\neg S$	modus tollens from (1), (2)
(4)	$\neg S \vee \neg D$	addition
(5)	$\neg(S \wedge D)$	DeMorgan
(6)	$(\neg R \vee \neg F) \rightarrow (S \wedge D)$	hypothesis
(7)	$\neg(\neg R \vee \neg F)$	moduls tollens from (5),(6)
(8)	$R \wedge F$	DeMorgan
(9)	R	simplification

q.e.d