CSE 321 Discrete Structures

Winter 2008
Lecture 17
Counting

Announcements

- Readings
  - Counting
    - 5.3, (4.3) Permutations and Combinations
    - 5.4, (4.4) Binomial Coefficients
    - 5.5, (4.5) Generalized Permutations and Combinations
  - Homework

Highlights from Lecture 16

- Counting
  - Product Rule
    - $|A_1 \times A_2| = |A_1||A_2|$
  - Sum Rule
    - A_1 and A_2 disjoint, $|A_1 \cup A_2| = |A_1| + |A_2|$
  - Inclusion-Exclusion
    - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Pigeon Hole Principle

Clever PHP Applications

- Every sequence of $n^2 + 1$ distinct numbers contains a subsequence of length n+1 that is either strictly increasing or strictly decreasing.

4, 22, 8, 15, 19, 11, 2, 1, 9, 20, 10, 7, 16, 3, 6, 5, 14

Proof

- Let $a_1, . . . a_m$ be a sequence of n^2+1 distinct numbers
- Let i_k be the length of the longest increasing sequence starting at a_k
- Let d_k be the length of the longest decreasing sequence starting at a_k
- Suppose $i_k \leq n$ and $d_k \leq n$ for all k
- There must be k and j, k < j, with $i_k = i_j$ and $d_k = d_j$

Permutations vs. Combinations

- How many ways are there of selecting 1st, 2nd, and 3rd place from a group of 10 sprinters?

- How many ways are there of selecting the top three finishers from a group of 10 sprinters?

r-Permutations

- An r-permutation is an ordered selection of r elements from a set
- P(n, r), number of r-permutations of an n element set

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-Combinations

- An r-combination is an unordered selection of r elements from a set (or just a subset of size r)
- C(r, n), number of r-permutations of an n element set

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

How many

- Binary strings of length 10 with 3 0's
- Binary strings of length 10 with 7 1's
- How many different ways of assigning 38 students to the 5 seats in the front of the class
- How many different ways of assigning 38 students to a table that seats 5 students

Prove C(n, r) = C(n, n-r) [Proof 1]

- Proof by formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

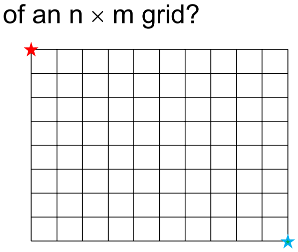
$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Prove C(n, r) = C(n, n-r) [Proof 2]

- Combinatorial proof

Counting paths

- How many paths are there of length n+m-2 from the upper left corner to the lower right corner of an n × m grid?



Binomial Theorem

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\end{aligned}$$

Binomial Coefficient Identities from the Binomial Theorem

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = (x+y)^n$$

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

$$\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n$$

Pascal's Identity and Triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\begin{array}{c}\binom{0}{0}\\ \binom{1}{0}\ \binom{1}{1}\\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}\\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}\\ \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}\\ \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}\\ \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}\\ \binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}\end{array}$$