CSE 321 Discrete Structures

Winter 2008

Lecture 24

Relations

Announcements

- Readings
  - Today
    - 8.3 Representing Relations
    - 8.4 Closures (Key idea – transitive closure)
    - 8.5 Equivalence Relations (Skim)
    - 8.6 Partial Orders
  - Next week
    - Graph theory

Highlights from Lecture 23

- Digraph representation of relations

- Matrix representation of relations

Matrix multiplication

Standard (×, +) matrix multiplication.
A is a m × n matrix, B is a n × p matrix
C = A × B is a m × p matrix defined:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

And-OR Matrix multiplication

A is a m × n boolean matrix, B is a n × p boolean matrix
C = A ⊗ B is a m × p matrix defined:

$$c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} (a_{11} \wedge b_{11}) \vee (a_{12} \wedge b_{21}) \vee (a_{13} \wedge b_{31}) & (a_{11} \wedge b_{12}) \vee (a_{12} \wedge b_{22}) \vee (a_{13} \wedge b_{32}) & \cdots \\ (a_{21} \wedge b_{11}) \vee (a_{22} \wedge b_{21}) \vee (a_{23} \wedge b_{31}) & (a_{21} \wedge b_{12}) \vee (a_{22} \wedge b_{22}) \vee (a_{23} \wedge b_{32}) & \cdots \\ (a_{31} \wedge b_{11}) \vee (a_{32} \wedge b_{21}) \vee (a_{33} \wedge b_{31}) & (a_{31} \wedge b_{12}) \vee (a_{32} \wedge b_{22}) \vee (a_{33} \wedge b_{32}) & \cdots \end{bmatrix}$$

Matrices and Composition

$$M_{S \circ R} = M_R \otimes M_S$$

R = {(a, a), (a, c), (b, a), (b, b)}
S = {(b, a), (b, c), (c, a), (c, c)}

Closures

- Reflexive Closure

- Symmetric Closure

Transitive Closure

- R = {(1, 2), (2, 3), (3, 4)}

Transitive closure

Equivalence Relations

Definition: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Are these equivalence relations?

- Congruence Mod m on $\mathbf{Z}^+$. R = {(a,b) | a ≡ b mod m}
- The 'divides' relation on $\mathbf{Z}^+$. R = {(a,b) | a|b}

Equivalence classes

- R = {(a,b) | a $\equiv$ b mod 3}, Domain: $\mathbf{Z}^+$

Partial Orderings

Definition: A relation R on a set S is called a *partial ordering* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*.

Are these posets?

- ($\mathbf{Z}$, $\geq$)
- ($\mathbf{Z}^+$, |)

Total Orderings

Definition: If (S, R) is a poset and every two elements of S are comparable, S is called a *totally (linearly) ordered set*, and R is called a *total (linear) order*.

Are these posets totally ordered?

- (**Z,** ≥)
- ($\mathbf{Z}^+$, |)