Discrete Structures

Probability

Chapter 6

Dieter Fox

Discrete Probability

♢ **Probability** : The probability of an event E, which is a subset of a finite sample space S of equally likely outcomes, is
$p(E) = |E|/|S|$.

♢ **Theorem:** Let E be an event in a sample space S. The probability of the event $\bar{E}$, the complementary event of E, is given by
$p(\bar{E}) = 1 - p(E)$.

♢ **Theorem:** Let E_1 and E_2 be events in a sample space S. Then
$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

Probability Theory

◇ Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign **probability $p(s)$ to each outcome s** . The following two conditions have to be met:

(i) $0 \leq p(s) \leq 1$ for each $s \epsilon S$

(ii) $\sum_{s \epsilon S} p(s) = 1$

◇ The **probability of the event E** is the sum of the probabilities of the outcomes in E. That is,

$p(E) = \sum_{s \epsilon E} p(s)$.

Conditional Probability

◊ Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

◊ The events E and F are said to be **independent** if and only if

$$p(E \cap F) = p(E)p(F).$$

Bernoulli Trial

- $\diamondsuit$ **Bernoulli Trial** : Experiment with only two possible outcomes: success or failure.

- $\diamondsuit$ **Probability of k successes in n independent Bernoulli trials** with probability of success p and probability of failure $q = 1 - p$, is $\binom{n}{k} p^k q^{n-k}$.

Random Variables

◇ A **random variable** is a function from the sample space of an experiment to the set of real numbers. That is a random variable assigns a real number to each possible outcome.

◇ The **distribution** of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r. A distribution is usually described by specifying $p(X = r)$ for each $r \in X(S)$.

Expectation of Random Variables

$\diamondsuit$ The **expected value** (or expectation) of a random variable $X(s)$ on the sample space S is equal to

$$E(X) = \sum_{s \epsilon S} p(s) X(s).$$

$\diamondsuit$ **Theorem** : If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r)\, r.$$

Expectation of Random Variables contd.

◇ **Theorem** : If X and Y are random variables on a space S, then
$E(X+Y) = E(X) + E(Y)$.
Furthermore, if $X_i, i = 1, 2, \ldots, n$, with n a positive integer, are random variables on S, and $X = X_1 + X_2 + \ldots + X_n$, then
$E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$.
Moreover, if a and b are real numbers, then $E(aX + b) = aE(X) + b$.

◇ **Theorem** : The expected number of successes when n Bernoulli trials are performed, where p is the probability of success on each trial, is np.

Independence

◇ The random variables X and Y on a sample space S are **independent** if for all real numbers r_1 and r_2

$$p(X(s) = r_1 \text{ and } Y(s) = r_2) \quad = \quad p(X(s) = r_1)\, p(Y(s) = r_2).$$

◇ **Theorem** : If X and Y are independent random variables on a space S, then $E(XY) = E(X)E(Y)$.

Variance

- Let X be random variables on a sample space S . The **variance** of X, denoted by $V(X)$, is

  $$V(X) = \sum_{s \epsilon S} (X(s) - E(X))^2 p(s).$$

  The **standard deviation** of X, denoted $\sigma(X)$, is defined to be $\sqrt{V(X)}$.

- **Theorem** : If X is a random variable on a space S, then

  $$V(X) = E(X^2) - E(X)^2.$$

- **Theorem** : If X and Y are two independent random variables on a space S, then $V(X+Y) = V(X) + V(Y)$. Furthermore, if $X_i, i = 1, 2, \ldots, n$ with n a positive integer, are pairwise random vairables on S, and $X = X_1 + X_2 + \ldots + X_n$,
  then $V(X_1 + X_2 + \ldots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n)$.