CSE 321 Discrete Structures

January 22, 2010

Lecture 08: Inductive Definitions

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $0 \in S$
  - Recursive step: if $x \in S$, then $x + 2 \in S$

What is the set S ?

- Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Terminology: “Recursive definition” = “Inductive Definition”

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $7 \in S$
  - Recursive step: if $x \in S$, $x \in S$, then $x - y \in S$
- Note: here we allow arbitrary integers, positive and negative

What is the set S ?

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $12 \in S$ and $21 \in S$
  - Recursive step: if $x \in S$, $x \in S$, then $x - y \in S$

What is the set S ?

Strings

The set Σ^* of strings over the alphabet Σ is defined as follows:

- Basis: $\lambda \in \Sigma^*$ (λ is the empty string)
- Recursive: if $w \in \Sigma^*$, $x \in \Sigma$, then $wx \in \Sigma^*$

Note: we sometimes write ε for the empty string

Strings

- Example: $\Sigma = \{a, b, c\}$. What is Σ^* ?

- $\Sigma^* = \{ \varepsilon, a, b, aa, ab, ba, bb, aaa, aab, . . .\}$

Families of strings over $\Sigma = \{a, b\}$

- L_1
  - $\lambda \in L_1$
  - $w \in L_1$ then $awb \in L_1$

- What is L_1 ?

Families of strings over $\Sigma = \{a, b\}$

- L_2
  - $\lambda \in L_2$
  - $w \in L_2$ then $aw \in L_2$
  - $w \in L_2$ then $wb \in L_2$

- What is L_2 ?

Families of strings over $\Sigma = \{a, b\}$

- Think of a as "(" and of b as ")"
- Define recursively the set L_3 of all well-formed parenthesis
- Strings that should be in L_3:
  - aaabbb, abababab, aabbabaaabbb, …
- Strings that should not be in L_3:
  - aab (too many a's), ba (unmatched), abbaab (unmatched)

Recursive Function definitions

The length of a string: **Len** : $\Sigma^* \rightarrow$ Int

$\textbf{Len}(\lambda) = 0;$

$\textbf{Len}(wx) = 1 + \textbf{Len}(w)$; for $w \in \Sigma^*$, $x \in \Sigma$

The concatenation of two strings: **Concat**: $\Sigma^* \times \Sigma^* \rightarrow \Sigma^*$

$\textbf{Concat}(w, \lambda) = w$ for $w \in \Sigma^*$

$\textbf{Concat}(w_1,w_2x) = \textbf{Concat}(w_1,w_2)x$ for w_1, w_2 in Σ^*, $x \in \Sigma$

Well Formed Fomulae

- $\Sigma = \{p, q, r, s, \ldots, T, F, \wedge, \vee, \rightarrow, \neg, (,)\}$
- Define Well-Formed-Formula for propositional logic
- Basis Step
  - p, q, r, s, … T, F are in WFF
- Recursive Step
  - If E and F are in WFF then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$ are in WFF

Well Formed Fomulae

Write recursive definitions on WFF for the following functions:

- Count the number of $\wedge$'s in the formula
- Test if a formula is *positive*, i.e. every atomic formula occurs under an even number of $\neg$ symbols (recall that $p \rightarrow q = \neg p \vee q$):

  $(\neg (\neg p \wedge \neg q)) \vee (s \wedge \neg\neg t)$ is positive

  $(\neg p \rightarrow s) \wedge t$ is positive

  $p \rightarrow s$ is not positive

Di-Graphs (Directed Graphs)

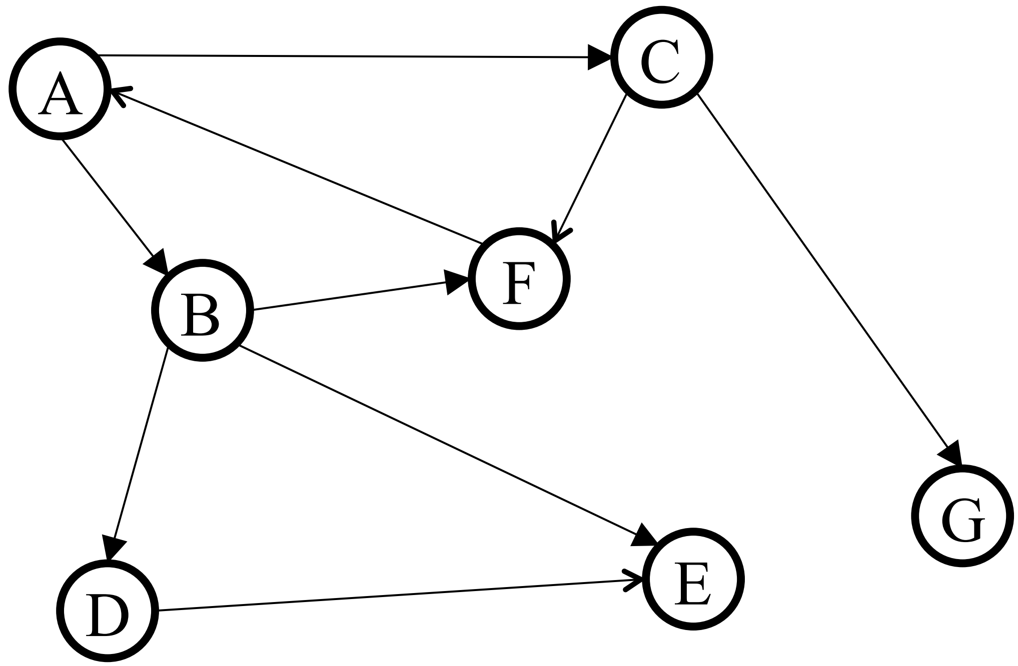
- Nodes: A,B,…
- Edges: A→B, …

- **Paths** from A to E:

  A,B,E

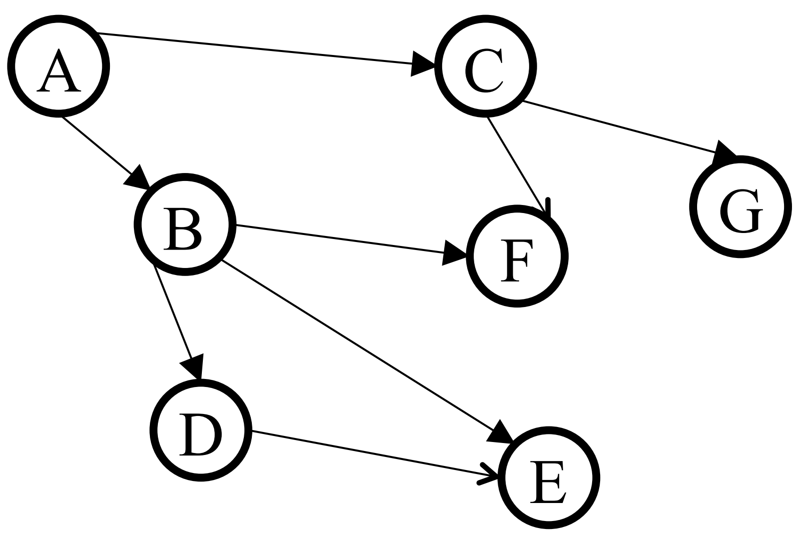
  A,B,D,E

  A,B,F,A,B,F,A,B,E
- **Cycle**: A,B,F,A

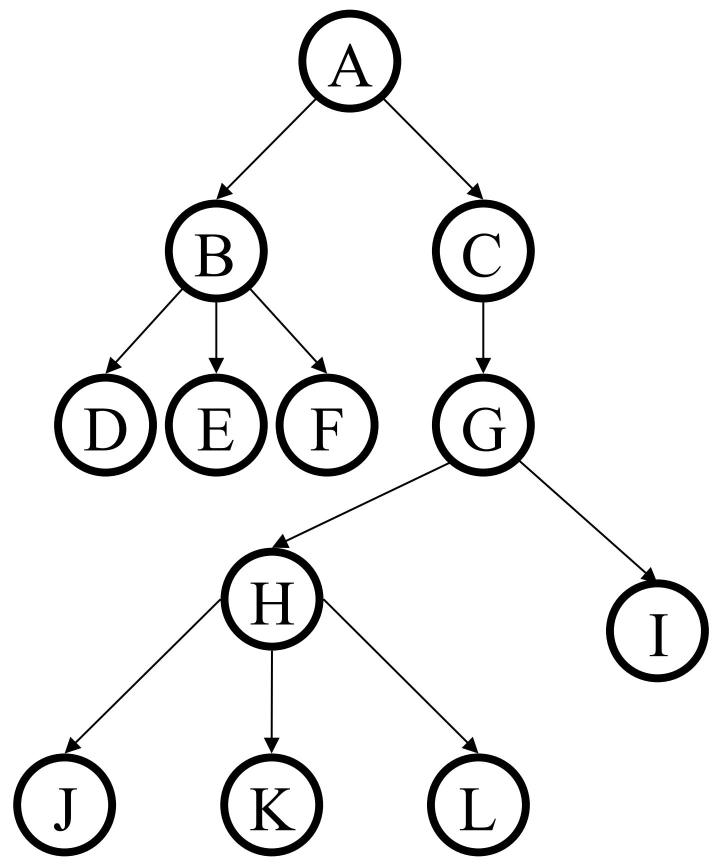


DAGs and Trees

A Directed Acyclic Graph (DAG) is a graph without cycles



A tree is like this:

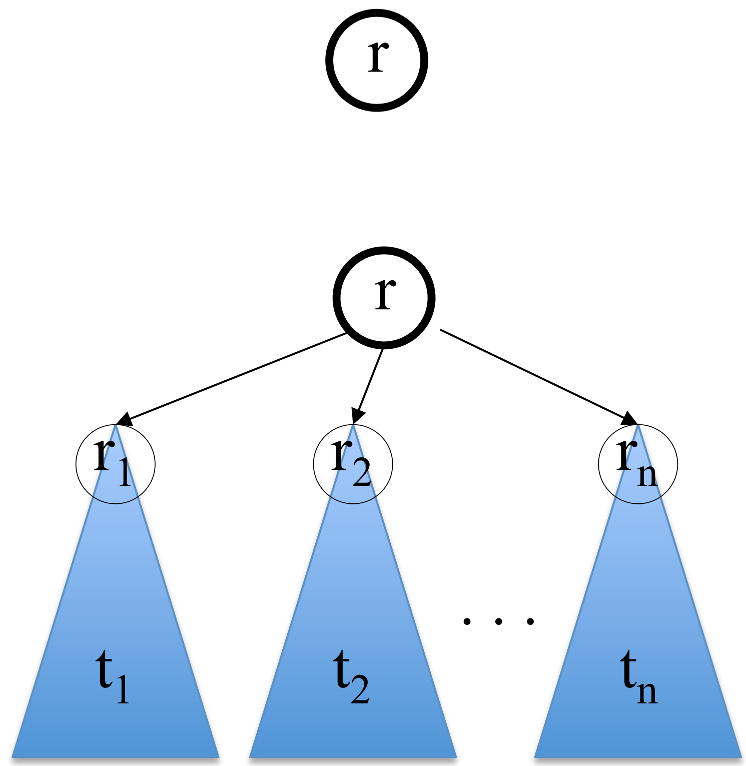


What is a “tree” ?

- “A tree is a graph such that….”
  - How would you define a tree ?
  - Want a tree to have a distinguished node, called the “root”

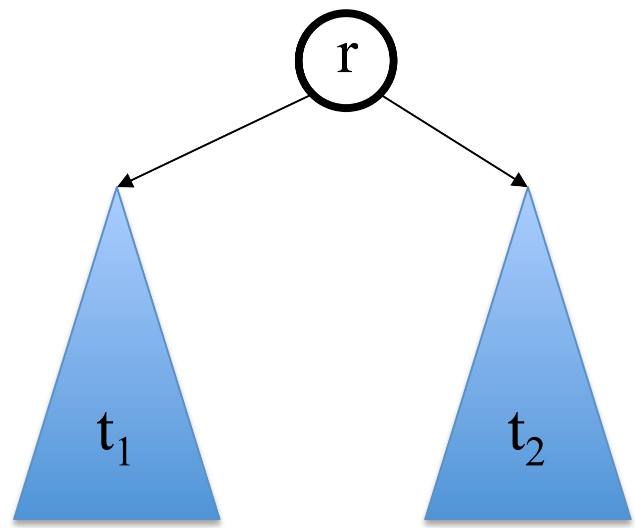
A Recursive Definition of Trees

- A graph with a single node r is a tree and its root is r
- If t_1, t_2, …, t_n are trees with roots r_1, r_2, …, r_n then the graph consisting of t_1, t_2, …, t_n, a new node r, and n edges (r, r_i), i=1,n, is a tree and its root is r.



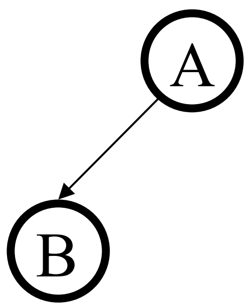
Extended Binary Trees

- The empty graph is an extended binary tree
- A nonempty extended binary tree has a root node r, with a left child t_1 and a right child t_2 s.t. both t_1 and t_2 are extended binary trees

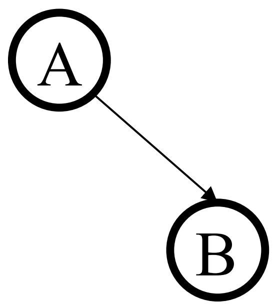


Subtle Distinction

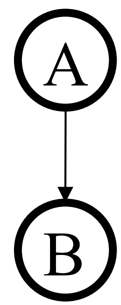
In an extended binary tree we distinguish between the left child and the right child:



Left child only



Right child only



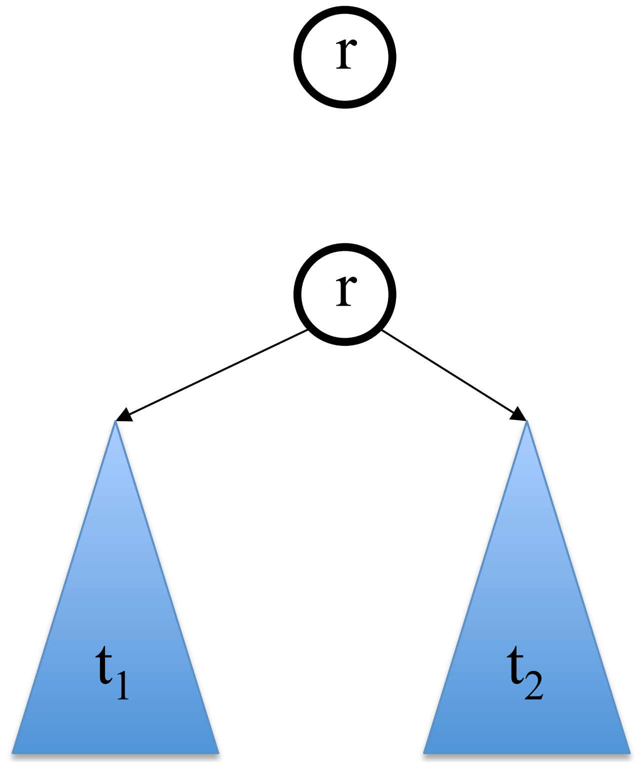
Not an “extended” binary tree

Full binary trees

- Now we want to rule out the empty trees and empty subtrees: “full binary tree”
- How do we do this ?

Extended Binary Trees

- The graph consisting of a single node is a full binary tree
- A nonempty full binary tree has a root node r, with a left child t_1 and a right child t_2 s.t. both t_1 and t_2 are full binary trees



Simplifying notation

- $(\bullet, T_1, T_2)$, tree with left subtree T_1 and right subtree T_2
- ε is the empty tree
- Extended Binary Trees (EBT)
  - $\varepsilon \in$ EBT
  - if $T_1, T_2 \in$ EBT, then $(\bullet, T_1, T_2) \in$ EBT
- Full Binary Trees (FBT)
  - $\bullet \in$ FBT
  - if $T_1, T_2 \in$ FBT, then $(\bullet, T_1, T_2) \in$ FBT

Recursive Functions on Trees

- $N(T)$ - number of vertices of T
- $N(\varepsilon) = 0$; $N(\bullet) = 1$
- $N(\bullet, T_1, T_2) = 1 + N(T_1) + N(T_2)$

- $Ht(T)$ – height of T
- $Ht(\varepsilon) = 0$; $Ht(\bullet) = 1$
- $Ht(\bullet, T_1, T_2) = 1 + \max(Ht(T_1), Ht(T_2))$

NOTE: Height definition differs from the text
Base case $H(\bullet) = 0$ used in text

More tree definitions: Fully balanced binary trees

- ε is a FBBT.
- if T_1 and T_2 are FBBTs, with $Ht(T_1) = Ht(T_2)$, then $(\bullet, T_1, T_2)$ is a FBBT.

And more trees: Almost balanced trees

- ε is a ABT.
- if T_1 and T_2 are ABTs with $Ht(T_1) -1 \leq Ht(T_2) \leq Ht(T_1)+1$ then $(\bullet, T_1, T_2)$ is a ABT.