CSE 321 Discrete Structures

February 17th, 2010

Lecture 17: Counting

Counting Rules

Product Rule: If there are n_1 choices for the first item and n_2 choices for the second item, then there are $n_1 n_2$ choices for the two items

Sum Rule: If there are n_1 choices of an element from S_1 and n_2 choices of an element from S_2 and $S_1 \cap S_2$ is empty, then there are $n_1 + n_2$ choices of an element from $S_1 \cup S_2$

Counting examples

License numbers have the form LLL DDD, how many different license numbers are available?

There are 38 students in a class, and 38 chairs, how many different seating arrangements are there if everyone shows up?

Important cases of the Product Rule

- Cartesian product

  $|A_1 \times A_2 \times \ldots \times A_n| = |A_1||A_2|. . . |A_n|$

- Subsets of a set S

  $|\boldsymbol{P}(S)| = 2^{|S|}$

- Strings of length n over Σ

  $|\Sigma^n| = |\Sigma|^n$

Counting Functions

Suppose |S| = n, |T| = m

How many functions from S to T?

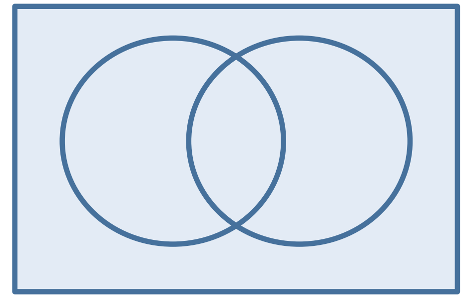
How many one-to-one functions from S to T?

More complicated counting examples

- BASIC variable names
  - Variables can be one or two characters long
    - The first character must be a letter
    - The second character can be a letter or a digit
    - The keywords “TO”, “IF”, and “DO” are excluded
- Passwords must be 4 to 6 characters long, and must contain at least one letter and at least one digit. (Case insensitive, no special characters)

Inclusion-Exclusion Principle

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

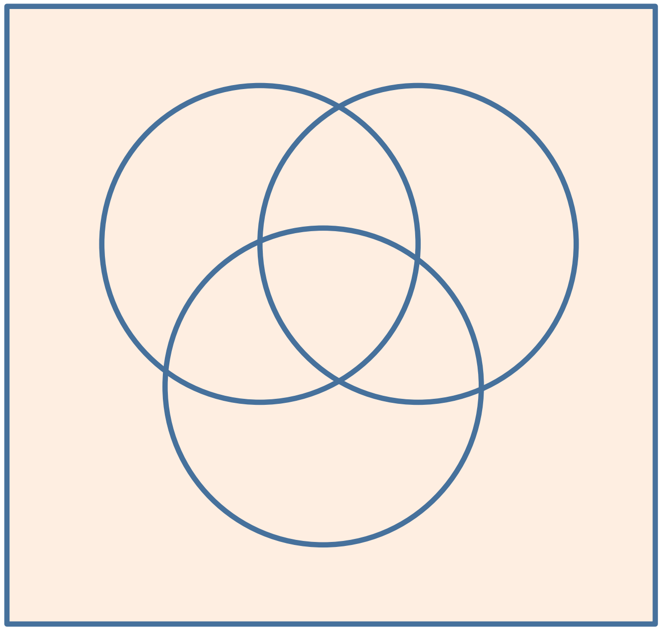


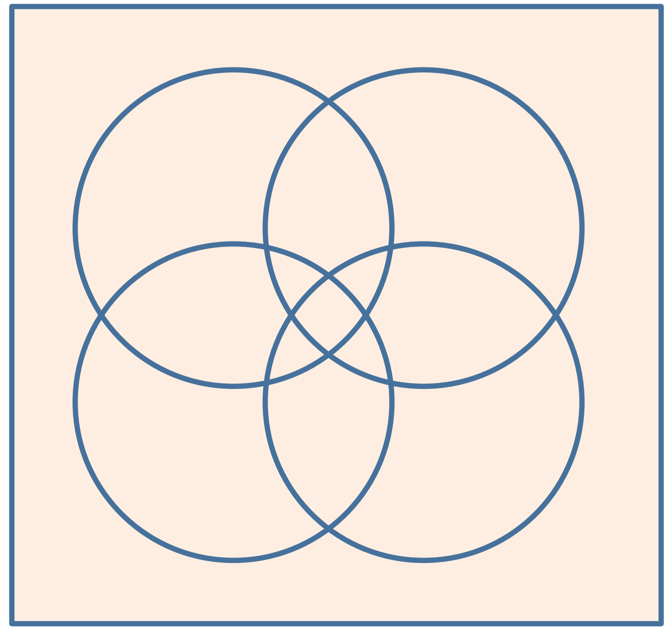
- How many binary strings of length 9 start with 00 or end with 11

Inclusion-Exclusion

- A class has of 40 students has 20 CS majors, 15 Math majors. 5 of these students are dual majors. How many students in the class are neither math, nor CS majors?

Generalizing Inclusion Exclusion





Pigeon Hole Principle

If k is a positive integer and k+1 or more objects are placed into k boxes, then at least one box has two or more objects

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects

Prove that if you have 800 people, at least three share a common birthday.

Clever PHP Applications

- Every sequence of $n^2 + 1$ distinct numbers contains a subsequence of length n+1 that is either strictly increasing or strictly decreasing.

$n=4, \; n^2+1 = 17$

4, 22, 8, 15, 19, 11, 2, 1, 9, 20, 10, 7, 16, 3, 6, 5, 14

This shows a decreasing subsequence of length 4
Note: there are also other increasing or decreasing subsequences

Proof

- Let $a_1, \ldots a_m$ be a sequence of n^2+1 distinct numbers
- Let i_k be the length of the longest increasing sequence starting at a_k
- Let d_k be the length of the longest decreasing sequence starting at a_k
- Suppose $i_k \leq n$ and $d_k \leq n$ for all k
- There must be k and j, $k < j$, with $i_k = i_j$ and $d_k = d_j$
- This is a contradiction:
  - if $a_k < a_l$ then $i_k > i_j$ (simply start at a_k continue with the longest increasing sequence starting at a_l)
  - if $a_k > a_l$ then $d_k > d_j$ (simply start at a_k continue with the longest decreasing sequence starting at a_l)

Permutations vs. Combinations

- How many ways are there of selecting 1st, 2nd, and 3rd place from a group of 10 sprinters?

- How many ways are there of selecting the top three finishers from a group of 10 sprinters?

r-Permutations

- An r-permutation is an ordered selection of r elements from a set
- P(n, r), number of r-permutations of an n element set

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-Combinations

- An r-combination is an unordered selection of r elements from a set (or just a subset of size r)
- C(r, n), number of r-combinations of an n element set

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

How many

- Binary strings of length 10 with 3 0's
- Binary strings of length 10 with 7 1's
- How many different ways of assigning 38 students to the first row in the class (the first row can seat 5 students)
- How many different ways of assigning 38 students to a table that seats 5 students

Prove C(n, r) = C(n, n-r) [Proof 1]

- Proof by formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

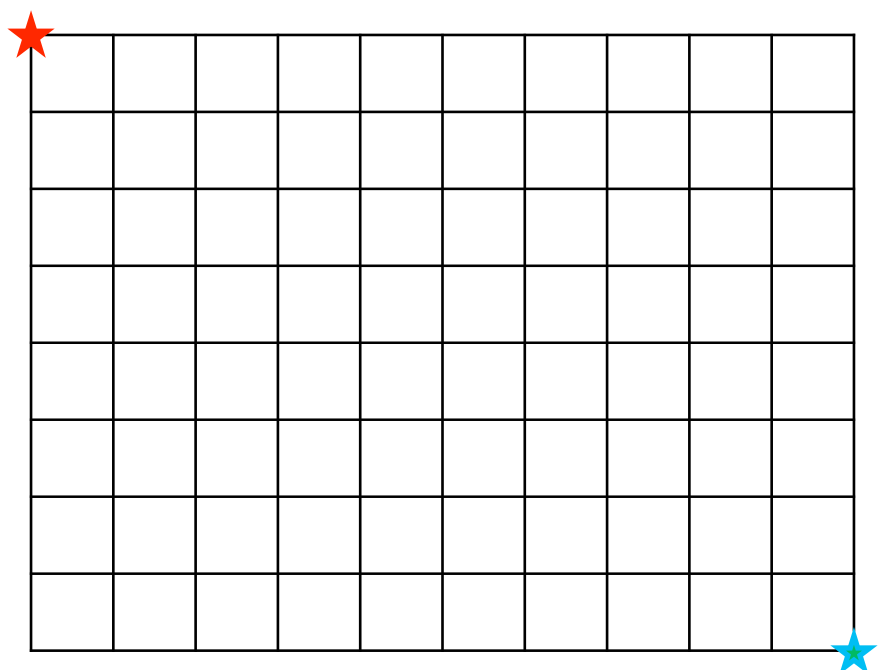
$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Prove C(n, r) = C(n, n-r) [Proof 2]

- Combinatorial proof

Counting paths

- How many paths are there of length n +m-2 from the upper left corner to the lower right corner of an $n \times m$ grid?



Binomial Theorem

$$
\begin{aligned}
(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n
\end{aligned}
$$

Binomial Coefficient Identities from the Binomial Theorem

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Pascal's Identity and Triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

Same thing, nicer:

C(n,k) = C(n-1,k-1) + C(n-1,k)

Recap

- Permutations

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

- Combinations

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

How many

- Let s_1 be a string of length n over Σ_1
- Let s_2 be a string of length m over Σ_2
- Assuming Σ_1 and Σ_2 are distinct, how many interleavings are there of s_1 and s_2?