CSE 321 Discrete Structures

March 3rd, 2010

Lecture 22: Applications of Probabilities

Expectation

The expected value of random variable X(s) on sample space S is:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

$$E(X) = \sum_{r \in X(S)} p(X = r) r$$

Flip a coin until the first head
Expected number of flips?

Random variable:

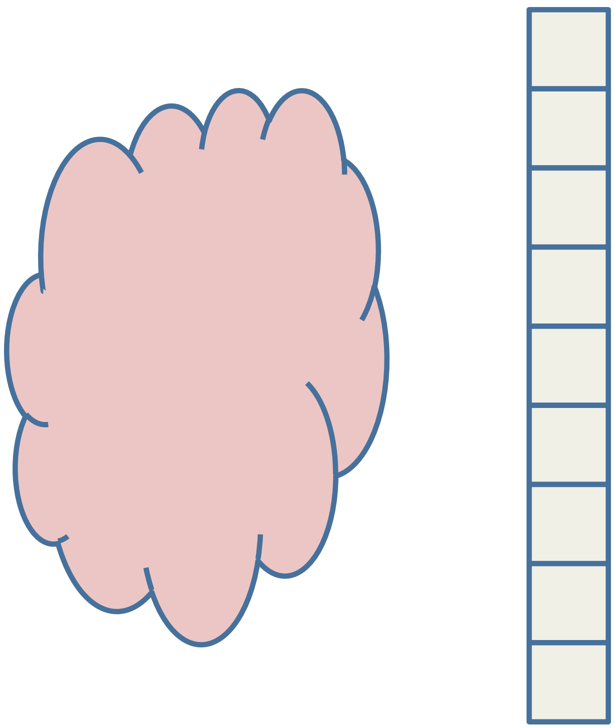
Computing the expectation:

Linearity of Expectation

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$E(aX) = aE(X)$$

Hashing



H: M $\rightarrow$ [0..n-1]

If k elements have been hashed to random locations, what is the expected number of elements in bucket j?

What is the expected number of collisions when hashing k elements to random locations?

Hashing analysis

Sample space: [0..n-1] × [0..n-1] × . . . × [0..n-1]

Random Variables
X_j = number of elements hashed to bucket j
C = total number of collisions

B_{ij} = 1 if element i hashed to bucket j
B_{ij} = 0 if element i is not hashed to bucket j

C_{ab} = 1 if element a is hashed to the same bucket as element b
C_{ab} = 0 if element a is hashed to a different bucket than element b

Counting inversions

Let $p_1, p_2, \ldots, p_n$ be a permutation of $1 \ldots n$

p_i, p_j is an inversion if $i < j$ and $p_i > p_j$

4, 2, 5, 1, 3

1, 6, 4, 3, 2, 5

7, 6, 5, 4, 3, 2, 1

Expected number of inversions for a random permutation

For each $i<j$ let X_{ij} be the following random variable:

$X_{ij} = 1$ if $p_i > p_j$, and $X_{ij} = 0$ if $p_i < p_j$

Fact 1: $P(X_{ij} = 1) = ½$ (why ??); hence $E(X_{ij}) = ½$

Let X = the number of permutations: $X = \Sigma_{ij}\ X_{ij}$

Fact 2: $E(X) = \Sigma_{ij}\ E(X_{ij})$; hence $E(X) = n(n-1)/4$

Insertion sort

```
for i :=1 to n-1{
    j := i;
    while (j > 0 and A[ j - 1 ] > A[ j ]){
        swap(A[ j -1], A[ j ]);
        j := j – 1;
    }
}
```

4	2	5	1	3

What is the expected number of swaps ?

Expected number of swaps for Insertion Sort

For each $i = 1, \ldots, n-1$, let X_i be the following random variable:

X_i = number of swaps at iteration i

Fact 1: Forall $j = 0, \ldots, i$, $P(X_i = j) = 1/(i+1)$ (why ??)

Fact 2: $E(X_i) = (1+2+\ldots+i)/(i+1) = i/2$

Let X be the total number of swaps

Fact 3: $E(X) = \Sigma_i \; E(X_i) = n(n+1)/4$

Left to right maxima

```
max_so_far := A[0];
k=0;
for i := 1 to n-1
    if (A[ i ] > max_so_far)
        { max_so_far := A[ i ];
          k++;
        }
return k;
```

5, 2, 9, 14, 11, 18, 7, 16, 1, 20, 3, 19, 10, 15, 4, 6, 17, 12, 8

What is the expected value of k ?

What is the expected number of left-to-right maxima in a random permutation