CSE373 Midterm I

Fall 2009

(Closed book, closed notes)

Name: ______________________________

Question	Points	Score
1	5	
2	10	
3	15	
4	10	
5	15	
6	10	
7	15	
8	5	
9	15	
Total:	100	

This exam is closed book and closed book. You will have 50 minutes. Justification or proof of your answer is rarely required but might lead to partial credit. Good luck!

1. (5 points) Show the truth table for the following propositional formula:

$$(p \vee \neg q) \wedge (q \vee \neg s) \wedge (s \vee \neg p)$$

Solution:

p	q	s	$(p \vee \neg q) \wedge (q \vee \neg s) \wedge (s \vee \neg p)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

2. (10 points) Which of these expressions are tautologies? No justifications are needed.

   (a) (2 points) $(p \to (r \to s)) \to ((p \to r) \to (p \to s))$

   (a) **T**

   (b) (2 points) $(p \vee q) \to (p \wedge q)$

   (b) **F**

   (c) (2 points) $(p \to (q \vee s)) \to ((p \to q) \vee (p \to s))$

   (c) **T**

   (d) (2 points) $((p \to s) \vee (q \to s)) \to ((p \vee q) \to s)$

   (d) **F**

   (e) (2 points) $\exists x.(P(x) \to \forall y.P(y))$

   (e) **T**

3. (15 points) Give a proof in natural deduction of the tautology

$$\vdash ((a \to b) \wedge (b \to c)) \to (a \to c)$$

Solution: To save space, I'll abbreviate:

$$A = (a \to b) \wedge (b \to c)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash a \to b}\wedge\mathrm{E} \quad \dfrac{}{A \vdash b \to c}\wedge\mathrm{E} \quad \dfrac{}{a \to b, a \vdash b}\to\mathrm{E} \quad \dfrac{}{b \to c, c \vdash c}\to\mathrm{E}}{A, a \vdash c}\textsc{Cut}}{A \vdash a \to c}\to\mathrm{I}}{\vdash A \to (a \to c)}\to\mathrm{I}$$

4. (10 points) For each sentence below indicate whether it is true or false, where A and B are Boolean formulas, and $\vdash$ denotes Classical Natural Deduction. You do not need to justify your answer.

   (a) (2 points) If p is a tautology, then p is satisfiable

   (a) **T**

   (b) (2 points) If p is satisfiable, then p is a tautology.

   (b) **F**

   (c) (2 points) If p is a tautology then $\vdash p$.

   (c) **T**

   (d) (2 points) If $\vdash p \to q$, then $p \vdash q$.

   (d) **T**

   (e) (2 points) If $\vdash p \vee q$ then either $\vdash p$ or $\vdash q$.

   (e) **F**

5. (15 points) Let $a[0], \ldots, a[n-1]$ be an array of integers. Write the following in predicate logic.

(a) (5 points) Every element of the array that is greater than or equal to 10 is equal to the sum of two other elements in the array.

Solution:

$$\forall i.\,(A[i] = 10 \rightarrow \exists j.\exists k.\,(A[j] + A[k] = A[i]))$$

(b) (5 points) All elements in the array are distinct.

Solution:

$$\forall i.\forall j. A[i] = A[j] \rightarrow i = j$$

(c) (5 points) The array is monotonically increasing.

Solution:

$$\forall i.\forall j. i < j \rightarrow A[i] < A[j]$$

6. (10 points) Prove the following by induction on n.

$$\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)=n$$

Solution: For $n=1$ we have the empty product

$$1=1$$

Now assume

$$\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)=n$$

and compute:

$$\begin{aligned}
\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right) &= n \\
\left(\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)\right)\left(1+\frac{1}{n}\right) &= n\left(1+\frac{1}{n}\right) \\
\sum_{i=1}^{n}\left(1+\frac{1}{i}\right) &= n+\frac{n}{n} \\
\sum_{i=1}^{n}\left(1+\frac{1}{i}\right) &= (n+1)
\end{aligned}$$

as desired.

7. (15 points) Find closed forms for the following sums. No justifications required, but show your work for possible partial credit.

   (a) (5 points)

$$\sum_{i=1}^{n}(2i-1)$$

(a) n^2

   (b) (5 points)

$$\sum_{i=1}^{\infty}\frac{1}{3^i}$$

(b) $\frac{1}{2}$

   (c) (5 points)

$$\sum_{i=1}^{n}(2i-1)^2$$

(c) $\frac{4n^3-n}{3}$

8. (5 points) Find a closed form for the following recurrence. No justification required, but show your work for possible partial credit.

$$\begin{aligned} x(0) &= 1 \\ x(n) &= 2x(n-1) + 1 \end{aligned}$$

8. $\underline{x(n) = 2^{n+1} - 1}$

9. (15 points) Consider the following function:

```
String f(int n) {
   if (n == 0) return "b"
   if (n == 1) return "aba"
   return f(n-2) + f(n-1) + f(n-2) + f(n-1) + f(n-2)
}
```

(a) (2 points) Compute $f(2)$ and $f(3)$.

Solution: $f(2) = b \cdot aba \cdot b \cdot aba \cdot b$
$f(3) = aba \cdot (b \cdot aba \cdot b \cdot aba \cdot b) \cdot aba \cdot (b \cdot aba \cdot b \cdot aba \cdot b) \cdot aba$

(b) (5 points) Let a_n be the number of a's returned by $f(n)$. Give a closed formula for a_n; your answer should be a function in n, e.g. $a_n = n^3 + 4^{2^n}$ (not the real answer).

Solution: $a_0 = 0$, $a_1 = 2$, $a_n = 2a_{n-1} + 3a_{n-2}$.
The characteristic equation is: $a^2 - 2a - 3 = 0$, $a_{1,2} = -1, 3$
$a_n = A \cdot (-1)^n + B \cdot 3^n$
$A + B = 0$, $-A + 3B = 2$, hence $A = -1/2, B = 1/2$.
Therefore $a_n = \frac{3^n - (-1)^n}{2}$.

(c) (5 points) Let b_n be the number of b's returned by $f(n)$. Give a closed formula for b_n.

> **Solution:** The characteristic equation is the same, hence:
> $b_n = A \cdot (-1)^n + B \cdot 3^n$
> The initial conditions are different:
> $A + B = 1$, $-A + 3B = 1$, hence $A = 1/2, B = 1/2$.
> Therefore $b_n = \frac{3^n+(-1)^n}{2}$.

(d) (3 points) Prove the following: when n is even then $f(n)$ has one more b than a; and when n is odd then $f(n)$ has one more a than b.

> **Solution:** Let's compute $b_n - a_n$:
>
> $$b_n - a_n = \frac{3^n + (-1)^n}{2} - \frac{3^n - (-1)^n}{2} = (-1)^n$$
>
> Thus, when n is even then $b_n - a_n = 1$. When n is odd then $b_n - a_n = -1$.