Solutions to Final Exam Sample Questions

CSE 321

1. Show that the proposition $p \to ((q \to (r \to s)) \to t)$ is a contingency WITHOUT constructing its full truth table.

   Solution: If p is false, then the proposition is true, because F implies anything. On the other hand, if q and t are false, then $((q \to (r \to s)) \to t)$ is false. Setting p to true makes the proposition $T \to F$ which is false. Since the proposition can be either true or false, it is a contingency.

2. Using the universe of integers for all quantifiers, express the following English sentences in predicate logic using the predicates P, E, and L where $P(x)$ means 'x is prime', $E(x)$ means 'x is even' and $L(x, y)$ means '$x < y$'.

   - No prime greater than 2 is even.
     Solution: $\forall x((P(x) \land L(2, x)) \to \neg E(x))$
   - For every number, there is a larger prime number.
     Solution: $\forall x \exists y(P(y) \land L(x, y))$

   Translate the following predicate logic expression into English where the predicates are the same as above: $\forall x((E(x) \land P(x)) \to L(x, 100))$

   Solution: "All even primes are less than 100."

3. In an undirected graph G, we define $\text{dist}(v, w)$ to be the length of the shortest path from v to w. Show that for vertices x, y, z, we have $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.

   Solution: Suppose that $\text{dist}(x, z) > \text{dist}(x, y) + \text{dist}(y, z)$, then $\text{dist}(x, z)$ would not be the length of the shortest path, because there is a path from x to z through y that has a distance of $\text{dist}(x, y) + \text{dist}(y, z)$ which is less than the supposed shortest length $\text{dist}(x, z)$. This is a contradiction. Thus it must be the case that $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.

4. Let $G = (V, E)$ be a directed graph. Define a relation R on E by (e_1, e_2) which is an element of R iff e_1 and e_2 lie on a common simple circuit. (Recall that a simple circuit is a path that starts and ends at the same vertex, and does not repeat any edges).

   - Is R necessarily reflexive? Prove or disprove.
     Solution: R is not reflexive. If G has no circuits, then there are no tuples (e_i, e_i) for any edge e_i.
   - Is R necessarily symmetric? Prove or disprove.
     Solution: R is symmetric. If e_i and e_j are on the same simple circuit, then it is necessarily the case that e_j and e_i are on the same simple circuit.
   - Is R necessarily transitive? Prove or disprove.
     Solution: R is not necessarily transitive. [INSERT PICTURE]

5. Let R be a relation on a set A. Is the transitive closure of R always equal to the transitive closure of R^2? Prove or disprove.

   Solution: Suppose $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$. Then $R^2 = \{(1, 3)\}$.

   Transitive closure of R is $R^* = \{(1, 2), (2, 3), (1, 3)\}$.

   Transitive closure of R^2 is $\{(1, 3)\}$.

   They are not always equal.

6. Prove that in the graph below, there are exactly 2^n paths of length $2n$ between vertex A and vertex B for $n \geq 1$.

```
    *   *   *   *
   / \ / \ / \ / \ /   \
A *   *   *   *   * . . . * B
   \ / \ / \ / \ / \   /
    *   *   *   *
```

   (n diamond shapes in a row)

   Solution: [NOT YET DONE]

7. The formula $\neg(p \to \neg q) \to (r \to (\neg s \to t))$ is false for exactly one assignment to p, q, r, s, and t. Find this assignment WITHOUT constructing a truth table.

   Solution:

$$\begin{aligned}
\neg(p \to \neg q) \to (r \to (\neg s \to t)) &\equiv \neg(\neg p \vee \neg q) \to (r \to (s \vee t)) \\
&\equiv \neg(\neg p \vee \neg q) \to (\neg r \vee (s \vee t)) \\
&\equiv (\neg p \vee \neg q) \vee (\neg r \vee s \vee t) \\
&\equiv \neg p \vee \neg q \vee \neg r \vee s \vee t.
\end{aligned}$$

   Let $p = T, q = T, r = T, s = F, t = F$.

8. Determine the number of strings over $\{a, b, c\}$ of length 100 that have exactly 98 a's.

   Solution: $\binom{100}{2} \cdot 2 \cdot 2$.

9. Suppose that you are working on a True-False test with 10 questions. Let p_i be the probability that you get answer i correct. Assume that the probabilities are independent.

   - What is the probability of getting all of the answers wrong (expressed as a formula involving p_1, p_2, ..., p_{10}).

     Solution: $\displaystyle\prod_{i=1}^{10}(1 - p_i)$

   - If $p_i = 1 - 2^{-i}$, what is your expected number of correct answers?

     Solution: Let X_i be 1 if question i is answered corrected, 0 otherwise. The expected number of correct answers is $E(X_1 + X_2 + ... + X_{10}) = E(X_1) + E(X_2) + ... + E(X_{10})$.

$$\begin{aligned}
E(X_i) &= 1 \cdot (1 - 2^{-i}) + 0 \cdot 2^{-i} \\
&= (1 - 2^{-i})
\end{aligned}$$

$$
\begin{aligned}
E(X_1 + X_2 + ... + X_{10}) &= E(X_1) + E(X_2) + ... + E(X_{10}) \\
&= (1 - 2^{-1}) + (1 - 2^{-2}) + ... + (1 - 2^{-10}) \\
&= 10 - (2^{-1} + 2^{-2} + ... + 2^{-10}) \\
&= 10 - (\frac{1}{2^1} + \frac{1}{2^2} + ... + \frac{1}{2^{10}}) \\
&= 10 - (\frac{2^9}{2^{10}} + \frac{2^8}{2^{10}} + ... + \frac{1}{2^{10}}) \\
&= 10 - \frac{1}{2^{10}}(2^9 + 2^8 + ... + 2^0) \\
&= 10 - \frac{1}{2^{10}}(2^{10} - 1) \\
&= 10 - \frac{1023}{1024} \\
&= 9\frac{1}{1024}
\end{aligned}
$$

10. Give an example of a relation which is not reflexive, not symmetric, not antisymmetric, and not transitive. (You are to give one relation that lacks all of these properties, not separate relations for each property.) Justify your answer.

    Solution: Let $R = \{(1,2),(2,1),(2,3)\}$ be a relation on the set of integers. R is not reflexive, because $(2,2)$ is not in R. R is not symmetric, because $(2,3)$ is in R, but $(3,2)$ is not. R is not antisymmetric, because $(1,2)$ and $(2,1)$ are in R, but $2 \neq 1$. R is not transitive, because $(1,2)$ and $(2,3)$ are in R, but $(1,3)$ is not.

11. Give three different equivalence relations on the set $\{a,b,c,d\}$.

    Solution:

    $\{(a,a),(b,b),(c,c),(d,d)\}$

    $\{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$

    $\{(a,a),(b,b),(c,c),(d,d),(c,d),(d,c)\}$

12. Let G be a directed graph with n vertices. Show that if G has a path of length greater than n, then G has a cycle.

    Solution:

    Let the path be of length $m > n$. The path can be written as: $(x_0,x_1),(x_1,x_2),(x_2,x_3),...(x_{m-1},x_m)$. There are $m+1 > n+1$ vertices on the path. By the pigeonhole principle, at least 2 of those vertices are equal as the graph only has n vertices. Let the index of the first instance of the vertex in the path be i and the index of the second instance of the vertex be j, then there is a cycle starting and ending at that vertex with the path $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),...,(x_{j-1},x_j)$.

13. Give a pair of non-isomorphic undirected graphs each with six vertices where every vertex has degree exactly two.

    Solution: Graphs need not be connected. The first graph is a hexagon. The second graph is a pair of triangles.

14. The following terms are used in propositional calculus. Give a one sentence definition for each of the terms.

   - Contingency
     Solution: A proposition that can be either true or false.
   - Contrapositive
     Solution: The contrapositive of $p \to q$ is $\neg q \to \neg p$.
   - Converse
     Solution: The converse of $p \to q$ is $q \to p$.

15. Give logical expressions for the following statements. Use quantifiers, connectives, and the predicates $P(x)$ and $H(x)$ which mean "x passed the class" and "x turned in all of the homework".

   - Every student that passed the class turned in all of the homework.
     Solution: $\forall x(P(x) \to H(x))$
   - There was a student that passed the class, but did not turn in all of the homework.
     Solution: $\exists x(P(x) \land \neg H(x))$

16. Consider the English alphabet, which consists of 5 vowels and 21 consonants.

   - How many strings of length 10 consist of 5 vowels followed by 5 consonants?
     Solution: $5^5 \cdot 21^5$
   - How many strings of length 10 consist of 5 distinct vowels followed by 5 distinct consonants. (In other words, the string has no repeated characters.)
     Solution: $P(5,5) \cdot P(21,5)$
     You may express your answers as products of integers.

17. Suppose that the propositions $p_1, p_2, ..., p_{n+1}$ are independently assigned random truth values, where the probability p_i is true is $\frac{1}{2}$.

   - What is the probability that the compound proposition $p_1 \land p_2 \land ... \land p_n$ is true?
     Solution: $(\frac{1}{2})^n$
   - What is the probability that the compound proposition $(p_1 \land p_2 \land ... \land p_n) \to p_{n+1}$ is true?
     Solution: $1 - p(\text{proposition is false}) = 1 - (\frac{1}{2})^{n+1}$

18. Let $A = \{x, y, z\}, B = \{1, 2, 3, 4\}$, and $C = \{a, b, c\}$. Let R be the following relation from A to B: $R = \{(x,1), (x,2), (y,2), (y,3), (z,3)\}$, and S be the following relation from B to C: $S = \{(1,a), (2,a), (2,b), (3,b), (4,b), (4,c)\}$. Compute the composition of R and S.

   Solution: $\{(x,a), (x,b), (y,a), (y,b), (z,b)\}$

19. Suppose R_1 and R_2 are transitive relations on a set A. Is the relation $R_1 \cup R_2$ necessariy a transitive relation? Justify your answer.

   Solution: No. $\{(1,2)\}$ and $\{(2,3)\}$ are each transitive relations, but their union $\{(1,2), (2,3)\}$ is not transitive.

20. Suppose R is the relation on the integers where xRy if and only if $x = y + 1$. Describe the relation that is the transitive closure of R.

   Solution: $R^* = \{(x,y) \mid x > y\}$

21. Let the relation L be defined on the integers by xLy if $|x| < |y|$ or $(|x| = |y| and x \leq y)$.

- List the elements of $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ under the L ordering.
- Argue that this ordering relation is total.

Solution: We did not cover this topic.

22. Let $G = (V, E)$ be the undirected graph that has as its vertices the set of integers, and edges: $E = \{(x, x+3) \mid x \text{ is an integer}\}$. Describe the connected components of G.

Solution: The graph is divided into three disjoint infinitely long graphs. Each vertex v is connected to two other vertices whose indices are the next larger (and smaller) number that is congruent to v modulo 3.

23. How many edges do the following undirected graphs contain.

- K_n, the complete graph on n vertices.
  Solution: $\binom{n}{2}$
- $K_{n,m}$, the complete bipartite graph between n vertices and m vertices.
  Solution: $n \cdot m$
- Q_n, the n dimensional hypercube. (Q_n has 2^n vertices.)
  Solution: We did not cover this topic.

24. Translate the following English sentences into predicate logic where the universe is the set of people and the allowable predicates are:

$S(x)$: x is a student

$F(x, y)$: x and y are friends

$O(x, y)$: x is older than y

- Every student has a friend who is also a student.
  Solution: $\forall x(S(x) \rightarrow \exists y(S(y) \wedge F(x, y)))$
- There is someone who is older than all of his/her friends.
  Solution: $\exists x \forall y(F(x, y) \rightarrow O(x, y))$

Write a predicate logic statement equivalent to the negation of each of the statements above that DOES NOT USE negation anywhere except immediately in front of the predicate symbols S, F, and O.

- **Solution:** $\exists x(S(x) \wedge \forall y(\neg S(y) \vee \neg F(x, y)))$
- **Solution:** $\forall x \exists y(F(x, y) \wedge \neg O(x, y))$

25. Suppose that $n \geq 3$ and $p_1, ..., p_n$ are propositions that are true or false independently with probability exactly $\frac{1}{2}$. Justify your answers.

- What is the probability that $(p_1 \vee p_2) \rightarrow p_3$ is true?
  Solution: $1 - p(\text{proposition is false}) = 1 - \frac{3}{4} \cdot \frac{1}{2} = \frac{5}{8}$
- What is the probability that $(p_1 \vee ... \vee p_{n-1}) \rightarrow p_n$ is true?
  Solution: $1 - p(\text{proposition is false}) = 1 - (1 - (\frac{1}{2})^{n-1}) \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2^n}$
- What is the expected number of propositions p_i that are true?
  Solution: $\frac{n}{2}$

26. In each of the following questions a relation R is defined. Assume that each R is defined over some natural set A. For each of the following, circle ALL properties that ALWAYS apply and X out those properties that do NOT apply at least some of the time.

    [R] Reflexive

    [S] Symmetric

    [A] Antisymmetric

    [T] Transitive

    You do not need to justify your answers.

    - xRy iff there is a path of length at least 0 from vertex v to vertex y in undirected graph G.
      Solution: [R], [S], [T]
    - xRy iff there is a path of length at least 0 from vertex v to vertex y in directed graph G.
      Solution: [R], [T]
    - The relation R defined on undirected graphs where (G_1, G_2) is in R iff G_1 is isomorphic to G_2.
      Solution: [R], [S], [T]
    - The relation R defined on the set of non-negative real numbers where xRy if and only if $x \leq y$.
      Solution: [R], [A], [T]
    - The relation R defined on the natural numbers where xRy iff $x = y \pmod{51}$.
      Solution: [R], [S], [T]
    - The relation R defined on the natural numbers where xRy iff $x = (y + 1) \pmod{51}$.
      Solution: [A]
    - The relation R defined on the set of people where xRy iff x is a grandparent of y.
      Solution: [A]
    - xRy iff $x = y$.
      Solution: [R], [S], [A], [T]
    - xRy iff x is a subset of y.
      Solution: [R], [A], [T]
    - xRy iff positive integer x divides positive integer y
      Solution: [R], [A], [T]

27. Here is a recursive definition of a "forest", a kind of undirected graph.

    - A graph with one or more nodes and no edges is a forest.
    - If $G = (V, E)$ is a forest then for any nodes x and y of G that are not connected by a path in G, the graph H consisting of G plus the edge (x, y) is also a forest.
    - (A graph is a forest only if it can be constructed using the above two rules.)

    Prove by induction that every forest with v vertices and e edges has exactly $v - e$ connected components.

    Solution:

    Basis step: A forest with no edges has v connected components (one per vertex). The forest has $v - e = v - 0 = v$ connected components.

    Recursive step: If G is a forest, then it has $v_G - e_G$ connected components. To form the forest H, we add the edge (x, y) where x and y of G are not connected by a path in G. H has one less connected component than G: $v_G - e_G - 1$ connected components. H has one more edge than G: $e_G = e_H - 1$. Both graphs have the same number of vertices. Thus H has $v_G - e_G - 1 = v_H - (e_H - 1) - 1 = v_H - e_H$ connected components.

28. Suppose that R is a relation on a set A. Prove or disprove: If R^2 is reflexive then R must be reflexive.

    Solution: Let $A = \{1, 2\}$ and $R = \{(1, 2), (2, 1)\}$. Then $R^2 = \{(1, 1), (2, 2)\}$ which is reflexive, but R is not.

29. Let R be the relation $\{3, 2), (3, 4), (1, 3), (2, 1)\}$ defined on the set $\{1, 2, 3, 4, 5\}$.

    - Draw the graph of R.
      Solution: [INSERT DRAWING]
    - Draw the graph of the transitive closure of R.
      Solution: [INSERT DRAWING]
    - Draw the graph of the reflexive-transitive closure of R.
      Solution: [INSERT DRAWING]

30. Let n be a positive integer. A perfect matching on a set of $2n$ vertices is an undirected graph with n edges, such that each vertex has degree exactly 1. For example, there is one perfect matching on any set of two vertices (with edge set $\{\{1, 2\}\}$ if the vertices are $\{1, 2\}$), and three distinct perfect matchings on four vertices (with edge sets $\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$ if the vertices are $\{1, 2, 3, 4\}$). Prove by induction that the number of perfect matchings on $2n$ vertices is the product of the odd numbers less than $2n$ (so for $n = 2$ it is $1 \cdot 3$).

    Solution: [NOT YET DONE]

31. As part of its effort to make U.S. paper money harder to counterfeit, the U.S. Treasury has decided to redesign the 1, 5, 10, 20, and 50 dollar bills. However, people have grown tired of seeing boring presidents on the bills so the Treasury Department is looking for more appealing candidates that appear on these 5 different bills. They have narrowed the field of available candidates to seven actors $\{a_1, ..., a_7\}$, eight basketball players $\{b_1, ..., b_8\}$, six cartoon characters $\{c_1, ..., c_6\}$, and five disc jockeys $\{d_1, ..., d_5\}$. (These four sets are pairwise disjoint, so there are 26 candidates in all.) Please JUSTIFY your answers for each of the following questions about the possible choices the Treasury Department might make. (Answers in the form of arithmetic expressions involving only +, *, -, /, and ! are acceptable for full credit.)

    (a) In how many ways can the Treasury Department decide which five candidates appear on the bills, without worring what bill they will appear on? (An example choice is $\{a_3, a_6, c_3, c_8, d_4\}$.)
    Solution: $\binom{26}{5}$

    (b) In how many ways can the Treasury Department make the same choice as in part (a) if they are required to choose at least one candidate from each of the four categories?
    Solution: $\binom{8}{2}\binom{7}{1}\binom{6}{1}\binom{5}{1} + \binom{8}{1}\binom{7}{2}\binom{6}{1}\binom{5}{1} + \binom{8}{1}\binom{7}{1}\binom{6}{2}\binom{5}{1} + \binom{8}{1}\binom{7}{1}\binom{6}{1}\binom{5}{2}$

    (c) If the Treasury chooses the candidates for the five bills randomly, so that each sequence of five distinct candidates has equal probability, what is the probability of the assignment "c_2 on \$1, d_4 on \$5, b_5 on \$10, c_6 on \$20, and a_1 on \$50"?
    Solution: $\frac{1}{P(26,5)}$

    (d) If the Treasury Department chooses the candidate randomly as in part (c), what is the probability that cartoon characters apear on the \$5 and \$10 bills but not on the \$50 bill?

    Solution: We start by picking candidates for the \$5, \$10, and \$50 bills. There are $6 \cdot 5 \cdot 20$ ways to do that. With the remaining 23 candidates, there are $23 \cdot 22$ ways to pick candidates for the \$1 and \$20 bills. The probability is thus $\frac{6 \cdot 5 \cdot 20 \cdot 23 \cdot 22}{P(26,5)}$.

32. - How many different undirected graphs are there on vertices $\{1, 2, 3\}$? Explain your answer.
      Solution: A complete graph on the 3 vertices has $\binom{3}{2} = 3$ edges. Any graph involving these 3 vertices would have a subset of those edges. There are $2^3 = 8$ possible subsets from having no edges to having all edges.

- How many of these are connected?

  Solution: With 3 vertices, a disconnected graph has either 0 edges or 1 edge. There's only one way to have 0 edges and there's $\binom{3}{2} = 3$ ways to choose 2 vertices for a single edge. There are 4 graphs that are disconnected, leaving 4 that are not.

- How many have exactly two connected components?

  Solution: The number of graphs with exactly two connected components is the number of graphs with exactly one edge, which is 3 as shown above.

- How many have Euler tours?

  Solution: To have an Euler tour, every vertex must have even degree. There are two ways, either every vertex has no edges, or every vertex has a degree of 2 making the complete graph.

- How many *non-isomorphic* undirected graphs are there on 3 vertices? (That is, if G and G' are isomorphic then we only count one of them.) Draw one of each.

  Solution: There are 4 types: graphs with no edges, 1 edge, 2 edges, and 3 edges.

33. To qualify for championships in certain games or sports there are sometimes "round-robin tournaments" in which each team plays exactly one game against each other team. Suppose that $n \geq 2$ teams played a round-robin tournament and assume that there were no ties (i.e. each game had a winner.)

    (a) If we define the relation R on the set of teams by xRy if and only if x won its game against y which of the following properties does R always have: Reflexive, Irreflexive, Symmetric, Antisymmetric, Transitive? (EXPLAIN why or why not for each of the 5 cases.)

    Solution:

    Reflexive: No, because teams do not play against themselves.

    Irreflexive: Yes, because teams do not play against themselves.

    Symmetric: No. If team x beats team y, then y cannot beat team x as they only play one game together.

    Antisymmetric: Yes, since (x, y) and (y, x) are never in R.

    Transitive: No. It is possible that x beats y, and y beats z, but z beats x.

    (b) Consider the directed graph G_R representing relation R from part (a). What is the sum of the in-degrees of all the vertices in G_R? What is the sum of the out-degrees? Explain your answers.

    Solution: The sum of the in-degrees of all vertices in a directed graph is equal to the sum of all the out-degrees which is equal to the number of edges. The directed graph has exactly one edge between every possible pair of vertices, because every team plays every other team and there is only one outcome. The number of edges if $\binom{n}{2}$ where n is the number of teams.

    (c) Prove that there is a team that won at least $\frac{n-1}{2}$ games where $[x]$ means the smallest integer $\geq x$.

    Solution: This question appears to be incomplete.

34. In stud poker (as opposed to draw poker), 4 of the 5 cards in the poker hand are dealt 'face up' so that everyone can see them and one card is 'face down' so nobody can see it. Suppose that we start with a perfectly shuffled deck so that the cards are equally likely to be in any order. EXPLAIN your answers to each of the following questions:

    - If only one hand is dealt, what is the probability that this is a 'full house' (recall that's 3 cards of one rank and 2 of another)?

      Solution: $\frac{P(13,2)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$

    - What is the conditional probability of that hand being a full house given that the four 'face up' cards are 2 pairs (i.e. 2 cards of each of two ranks)?

      Solution: $\frac{4}{48} = \frac{1}{12}$

- What is the conditional probability of that hand being a full house given that four 'face up' cards contain 3 of one rank and 1 of another?

  Solution: $\frac{3}{48} = \frac{1}{16}$

- If 4 players are dealt poker hands from this perfectly shuffled deck, what is the expected number of players who have full houses?

  Solution: [NOT YET DONE]

35. In the land of Garbanzo, the unit of currency is the bean. They only have two coins, one worth 2 beans and the other worth 5 beans.

    - Give a recursive definition of the set of positive integers S such that x is in S if and only if one can make up an amount worth x beans using at most one 5-bean coin and any number of 2-bean coins.

      Solution:

      Basis step: $0 \epsilon S$ and $5 \epsilon S$

      Recursive step: If $x \epsilon S$, then $(x+2) \epsilon S$.

    - Prove by strong induction that every integer ≥ 4 is in S. (You do NOT need a recursive proof here.)

      Solution: Let $P(n)$ be "$n \epsilon S$".

      Base cases: $P(4)$ is true, because $0 \epsilon S$ and thus applying the recursive step twice shows that $2 \epsilon S$ and $4 \epsilon S$. $P(5)$ is true by the basis step of the recursive definition.

      Inductive hypothesis: $P(4) \wedge P(5) \wedge ... \wedge P(k)$ is true.

      Inductive step: $P(k+1)$ is true, because $P(k-1)$ states that $(k-1) \epsilon S$ and applying the recursive step of the definition states that $((k-1)+2) \epsilon S$.

36. Let R be the relation $\{(1,2),(3,4),(1,3),(2,1)\}$ defined on the set $\{1,2,3,4,5\}$.

    - Draw the graph of R.

      Solution: [INSERT DRAWING]

    - Draw the graph of the transitive closure of R.

      Solution: [INSERT DRAWING]

    - Draw the graph of the reflexive-transitive closure of R.

      Solution: [INSERT DRAWING]

37. If R is a relation on a set A then we say that R is *total* if and only if $\forall x \epsilon A (\exists y \epsilon A (xRy))$. Prove that if R is symmetric and total then its transitive closure is also reflexive.

    Solution: For all $x \epsilon A$, there is a tuple $(x,y) \epsilon R$ for some y, because R is total. Since R is symmetric, $(y,x) \epsilon R$. From (x,y) and (y,x), the transitive closure will include (x,x) for all $x \epsilon A$ and is thus reflexive.

38. For each $n \geq 0$ define T_n, the 'complete 3-ary tree of height n' as follows:

    - T_0 is an undirected graph consisting of a single vertex called the root of T_0.
    - For $n \geq 0$, T_{n+1} is an undirected graph consisting of a new vertex of degree 3 joined to the roots of 3 disjoint copies of T_n. The new vertex is called the root of T_{n+1}.

    Prove by induction that for each $n \geq 0$, T_n has exactly $\frac{3^{n+1}-1}{2}$ vertices.

    Solution: Let $V(n)$ be "T_n has exactly $\frac{3^{n+1}-1}{2}$ vertices."

    Base case: $V(0)$ is true, because T_0 has $\frac{3^{0+1}-1}{2} = \frac{3-1}{2} = 1$ vertex.

Inductive hypothesis: $V(k)$ is true, i.e. "T_k has exactly $\frac{3^{k+1}-1}{2}$ vertices."

Inductive step: T_{k+1} has 3 disjoint copies of T_k, with each copy having $\frac{3^{k+1}-1}{2}$ vertices totalling $3 \cdot \frac{3^{k+1}-1}{2} = \frac{3^{(k+1)+1}-3}{2}$ vertices. Adding 1 for the new vertex of T_{k+1} results in $\frac{3^{(k+1)+1}-3}{2} + 1 = \frac{3^{(k+1)+1}-3}{2} + \frac{2}{2} = \frac{3^{(k+1)+1}-1}{2}$ vertices.

39. Suppose that you have a rectangular cake which you need to cut into n rectangles of equal size by successively cutting the cake into chunks. Show that no matter how you do it you need to make exactly $n-1$ cuts in the cake (where a single cut is a straight line cut of one chunk, i.e. you are NOT allowed to line up several chunks and cut them at once.) (Hint: induction may help.)

    Solution: Let $P(n)$ be "Cutting a cake into n rectangles of equal size requires $n-1$ cuts".

    Base case: $P(1)$ is true, because no cuts are necessary.

    Inductive hypothesis: $P(1) \wedge P(2) \wedge ... \wedge P(k)$ is true, i.e. cutting a cake into j rectangles of equal size, where $j \leq k$, requires $j-1$ cuts.

    Inductive step: When making the first cut, the area of both chunks must be multiples of $\frac{A}{k+1}$ where A is the area of the cake, otherwise the cake cannot be cut into $k+1$ rectangles of equal size in the end. Let the two chunks have area of $r\frac{A}{k+1}$ and $s\frac{A}{k+1}$ with $r+s=k+1$. We want to cut the first chunk into r pieces of equal size, in particular $\frac{A}{k+1}$. Since, $r \leq k$, we can apply the inductive hypothesis. Cutting the first chunk requires $r-1$ cuts. Similarly cutting the second chunk requires $s-1$ chunks. Including the first cut, the total number of cuts is $(r-1)+(s-1)+1=(r+s)-1=(k+1)-1$.