# CSE 321
## Solutions to Practice Problems

**Instructions:**

- Feel free NOT to multiply out binomial coefficients, factorials, etc, and feel free to leave answers in the form of a sum.
- No calculators, books or notes are allowed.

1. True or False:

    - $p \to q$ is logically equivalent to $q \to p$.
    False. Consider $p = true$ and $q = false$, then $p \to q$ is false and $q \to p$ is true.
    - $((p \to q) \wedge \neg p) \to \neg q$ is a tautology.
    False. Consider $p = false$ and $q = true$, then the above statement is false.
    - $((\forall x[P(x) \to Q(x)]) \wedge P(y)) \to Q(y)$ is a tautology.
    True.
    - There is a one-to-one function from $A$ to $B$ if and only if there exists an onto function from $B$ to $A$.
    True. There is a one-to-one function form $A$ to $B$ only if $|A| \leq |B|$, in which case there is a on-to function from $B$ to $A$.
    - To prove by contradiction that $p \to q$, one must show that $p$ is false.
    True. One must show assuming $q$ is false that $p$ is false, thus contradicting assumption that $p$ is true.
    - $Pr(A \bigcup B) \leq Pr(A) + Pr(B)$.
    True. Follows from $\Pr(A \bigcup B) \leq \Pr(A) + \Pr(B) - \Pr(A \cap B)$
    - For any event $A$ in a probability space $0 \leq Pr(A) \leq 1$.
    True.
    - For any events $A$ and $B$ in a probability space $Pr(A \mid B) = Pr(A)$.
    False. It is true only if $A$ and $B$ are independent.
    - An undirected graph has an even number of vertices of odd degree.
    True. This is because the sum of the degrees of vertices of a graph = twice the number of edges which is always an even number.

2. On the next set of questions, fill in the blanks.

    - If a set $A$ is contained in a set $B$, then $A \cup B =$ ............ B
    - If a set $A$ is contained in a set $B$, then $A \cap B =$ ............ A
    - The number of subsets of an $n$ element set is ............ $2^n$

- The number of ways of choosing an unordered subset of size $k$ out of a set of size $r$ is ............. $\binom{r}{k}$
- The coefficient of $x^{10}$ in the polynomial $(5x+1)^{100}$ is ............. $\binom{100}{10}5^{10}$
- The number of different binary relations from a set $A$ of size $n$ to a set $B$ of size $m$ is ............. $2^{nm}$
- The number of different reflexive binary relations on a set $A$ of size $n$ is ............. $2^{n^2-n}$
- The number of different undirected graphs (no self loops and no parallel edges) on $n$ vertices is ............. $2^{\frac{n(n-1)}{2}}$
- What is the coefficient of $x^7$ in $(10x+2)^{21}$? $\binom{21}{7}10^7 2^{14}$
- What is the probability of getting exactly 12 heads if a biased coin with probability 4/5 of coming up heads is tossed 25 times (independently)? $\binom{25}{12}(\frac{4}{5})^{12}(\frac{1}{5})^{13}$

3.

- Every day, starting on day 0, one vampire arrives in Seattle from Transylvania and, starting on the day after its arrival, bites one Seattlite every day. People bitten become vampires themselves and live forever. New vampires also bite one person each day starting the next day after they were bitten. Let $V_n$ be the number of vampires in Seattle on day $n$. So, for example, $V_0 = 1$, $V_1 = 3$ (one that arrived from Transylvania on day 0, one that he bit on day 1, and another one that arrived from Transylvania on day 1), $V_2 = 7$ and so on. Write a recurrence relation for $V_n$ that is valid for any $n \geq 2$.

  **Solution:** If there are $V_{n-1}$ vampires on day $n-1$, they bite $V_{n-1}$ on day $n$. Further an additional vampire arrives from Transylvania on day $n$. Therefore the total number of vampires on day $n$ is given by

$$V_n = V_{n-1} + V_{n-1} + 1 = 2V_{n-1} + 1$$

- Prove by induction on $n$ that $V_n \leq 3^n$.

  **Solution:** To prove $V_n \leq 3^n$ by induction.
  Base Case: For $(n = 0)$, $V_0 = 1 \leq 3^0$.
  I.H: For $n = k$, we know that $V_k \leq 3^k$.
  To show for $n = k+1$,

$$V_{k+1} = 2 \cdot V_k + 1$$

  Using induction hypothesis, we get

$$\begin{aligned} V_{k+1} &= 2 \cdot V_k + 1 \\ &\leq 2 \cdot 3^k + 1 \\ &\leq 2 \cdot 3^k + 3^k \\ &\leq 3^{k+1} \end{aligned}$$

  Hence by principle of mathematical induction, we conclude that for all $n$, $V_n \leq 3^n$

4. (25 points) Consider an exam consisting of 25 True/False questions. Suppose that a student has probability 1/2 of getting the answer to a particular question right, independently for all questions.

   (a) In how many different ways can the student answer the questions?
   **Solution:** $2^{25}$. Since there are two choices for each question.

   (b) What is the probability that the student answers the second question correctly given that the student answers the first question correctly?
   **Solution:** $\frac{1}{2}$. As the two choices are independent of each other.

   (c) What is the probability that the student answers the first two questions correctly given that the student answers *at least* one of the first two questions correctly?
   **Solution:** $\frac{1}{3}$. Because, given that the student answers atleast one question right, there are three possibilities $(correct, incorrect)$ $(incorrect, correct)$ $(correct, correct)$ all of which are equally likely.

   (d) What is the expected number of answers the student gets right? Briefly explain your answer.
   **Solution:**Define indicator random variables $X_1 \dots X_{25}$, where the variable $X_i$ is 1 if the student gets the $i^{th}$ question right, and 0 otherwise. The total number of questions that student gets right is given by $Y = \sum_{i=1}^{25} X_i$. By linearity of expectation we can write:

$$E[Y] = \sum_{i=1}^{25} E[X_i]$$

   Further for any $i$,

$$E[X_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$

   Substituting the value of $E[X_i]$ in the original equation, we get

$$E[Y] = 25 \cdot \frac{1}{2} = \frac{25}{2}$$

   (e) What is the expected number of points the student gets on the exam if the student gets 2 points for each question answered correctly and gets 1 point taken away (or equivalently -1 point) for each question answered incorrectly?
   **Solution:**Define random variables $S_1 \dots S_{25}$, where the variable $S_i$ is the score, the student gets on the $i^{th}$ question. The total score of the student is given by $Y = \sum_{i=1}^{25} S_i$. By linearity of expectation we can write:

$$E[Y] = \sum_{i=1}^{25} E[S_i]$$

   Further for any $i$,

$$E[S_i] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-1) = \frac{1}{2}$$

Substituting the value of $E[S_i]$ in the original equation, we get

$$E[Y] = 25 \cdot \frac{1}{2} = \frac{25}{2}$$

5. 
- How many permutations of the letters {a,b,c,d,e,f,g,h} are there?
  **Solution:**$8! = 8 \times 7 \times 6 \ldots \times 1$
- How many permutations of {a,b,c,d,e,f,g,h} are there that don't contain the letters "bad" (appearing consecutively)?

  **Solution:**To compute the number of permutations that contain, we can consider "bad" as another alphabet. So we have {c,e,f,g,h,bad} as our alphabet. Therefore the number of permutations which contain "bad" is given by 6!. Therefore, number of permutations that do not contain "bad" is $8! - 6!$
- How many permutations of {a,b,c,d,e,f,g,h} are there that don't contain either the letters "bad" appearing consecutively or the letters "fech" appearing consecutively?

  **Solution:**Let $S_{bad}$ denote the set of permutations that have "bad" occuring in them. Similarly let $S_{fech}$ denote the set of permutations that have "fech" occuring in them.

  $$|S_{bad} \cup S_{fech}| = |S_{bad}| + |S_{fech}| - |S_{bad} \cap S_{fech}|$$

  The number of permutations that contain "bad" ($|S_{bad}|$) as we computed in the previous problem is 6!. Similarly considering "fech" as an alphabet, we get that the number of permutations that contain "fech" is given by 5!. To compute permutations that contain both "bad" and "fech", we need to look at permutations of the set {g,bad,fech}. Therefore, the number of permutations that contain both "bad" and "fech" is 3!. Substituting in the above formula we get

  $$|S_{bad} \cup S_{fech}| = 6! + 5! - 3!$$

  We are interested in finding the number of permutations that do not contain either "bad" or "fech". This is given by

  $$\text{Number of permutations without "bad" or "fech"} = 8! - (6! + 5! - 3!)$$
- How many words of length 10 can be constructed using the letters {a,b,c,d,e,f,g,h} that contain exactly 3 a's? (They don't have to have any English meaning.)

  **Solution:**Let us choose the three locations where 'a' occurs, this can be done in $\binom{10}{3}$ ways. After filling out the 'a's we have 7 choices for each of the remaining 7 locations in the word. Therefore the total number of words that contain exactly 3 a's is $\binom{10}{3} \times 7^7$.

6. Suppose a biased coin with probability 3/4 of coming up heads is tossed independently 100 times.

- What is the conditional probability that the first 50 tosses are heads given that the total number of heads is 50?

  **Solution:**Let $A$ be the event that there are 50 heads. $B$ be the event that there are 50 heads in the beginning. We need to compute $\Pr(B|A)$

$$\Pr(A) = \binom{100}{50}(\frac{3}{4})^{50}(\frac{1}{4})^{50}$$

  In order to compute that $A \cap B$, we need to compute the probability that there are 50 heads in the beginning and that there are 50 heads in total. That is the probability that there are 50 heads, followed by 50 tails.

$$\Pr(A \cap B) = (\frac{3}{4})^{50}(\frac{1}{4})^{50}$$

  Therefore we get

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1}{\binom{100}{50}}$$

- What is the expected number of heads?

  **Solution:**Define indicator random variables $X_1 \ldots X_{100}$, where the variable $X_i$ is 1 if the $i^{th}$ toss is head, and 0 otherwise. The total number of heads is given by $Y = \sum_{i=1}^{100} X_i$. By linearity of expectation we can write:

$$E[Y] = \sum_{i=1}^{100} E[X_i]$$

  Further for any $i$,

$$E[X_i] = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$$

  Substituting the value of $E[X_i]$ in the original equation, we get

$$E[Y] = 100 \cdot \frac{3}{4} = 75$$

- Suppose that you are paid \$50 if the number of heads in the first **two** tosses is even and \$100 if the number of heads in these first two tosses is odd. What is your expected return?

  **Solution:**The probability that the number of heads in first two tosses is odd is :

$$\Pr(\text{odd}) = \Pr(HT) + \Pr(TH) = \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{8}$$

$$\Pr(\text{even}) = 1 - \Pr(\text{odd}) = \frac{5}{8}$$

  Therefore the expected return is given by

$$\text{Expected Return} = \frac{5}{8} \cdot 50 + \frac{3}{8} \cdot 100 = \frac{550}{8}$$

7. Use the Euclidean algorithm to find gcd (486, 446).

   **Solution:**

$$\begin{aligned} gcd(486,446) &= gcd(446,40) & 486 = 446\cdot 1+40 \\ &= gcd(40,6) & 40 = 6\cdot 6+4 \\ &= gcd(6,4) & 6 = 4\cdot 1+2 \\ &= gcd(4,2) & 4 = 2\cdot 2+0 \\ &= gcd(2,0) & \\ &= 2 & \end{aligned}$$

8. A lake contains $n$ trout. 100 of them are caught, tagged and returned to the lake. Later another set of 100 trout are caught, selected independently from the first 100.

   - Write an expression (in $n$) for the probability that of the second 100 trout caught, there are exactly 7 tagged ones.
     **Solution:** $P(N=7) = [C(100,7)C(n-100,93)]/C(n,100)$
   - Now consider selecting the second 100 trout with replacement. That is, you repeat 100 times the following steps: select at random a trout in the lake, check if the trout is tagged and return it to the lake before selecting a new trout. What is the probability that exactly 7 of the selected trout are tagged?
     **Solution:** $P(N=7) = [C(100,7)100^7(n-100)^{93}]/n^{100}$

9. Prove that any undirected, connected graph with $n$ vertices and no cycles (i.e., a tree) has exactly $n-1$ edges. (You should assume that the graph has no self loops and no parallel edges. A cycle is a sequence of 2 or more distinct edges that start and end at the same vertex.)
   **Solution:** Proof by induction
   Base Case: n=1, Graph G has 0 edge. Vertices num $|V| = 1$, edge num $|E| = 0$, so $|E| = |V| - 1$ holds.
   I-H: Assume that for any G containing k vertices, $|E| = |V| - 1$ holds, for all $1 \leq k \leq n$. (Strong Induction)
   To prove: For n+1, the conclusion also holds.
   Prove: Let $G$ be a graph with $n+1$ vertices. Remove a vertex $v$ and all its edges from $G$. Let us say on removing $v_{k+1}$, the graph $G$ splits in to $r$ pieces(connected components) $C_1, \ldots, C_r$. Let $a_1, a_2, \ldots, a_r$ denote the number of vertices in $C_1, \ldots, C_r$ respectively, then it is clear that

$$\sum_{i=1}^{r} a_i = n$$

   Observe, that since the original $G$ is connected, there is an edge from $v$ to each the components $C_i$. Further observe that there cannot be two edges from $v$ to a component, as then there would be a cycle. Therefore there is exactly one edge from $v$ to each of the components $C_i$. As $G$ does not have any cycle, any of the subcomponents $C_i$ also do not have cycles. Further by definition of $C_i$, each of them is a connected. Therefore applying the induction hypothesis on $C_i$, we conclude that the number of edges in $C_i$ is $a_i - 1$. Therefore the total number of edges in $G$ is given by

$$\begin{aligned}
\text{Total number of edges in } G &= \text{Number of edges incident on } v + \sum_i \text{Number of edges in } C_i \\
&= r + \sum_{i=1}^{r} (a_i - 1) \\
&= r + \sum_{i=1}^{r} a_i - r \\
&= \sum_{i=1} a_i \\
&= n \\
&= (n+1) - 1
\end{aligned}$$

Thus by induction the result is true for all $n$, and hence for all graphs.

10. Suppose that for all $n \geq 1$

$$g(n+1) = max_{1 \leq k \leq n}[g(k) + g(n+1-k) + 1]$$

and that $g(1) = 0$. Prove by induction that $g(n) = n - 1$ for all $n \geq 1$.
**Solution:** Base: n=1, g(1)=0=1-1, it holds.
n=2, $g(2) = max_{1 \leq k \leq 1}[g(k) + g(1+1-k) + 1] = 1$, it holds.
I-H: $g(k) = k - 1$ holds for $k \geq 1$.
To prove: $g(k+1) = k$.
Proof:
$g(k+1) = max_{1 \leq m \leq k}[g(m) + g(k+1-m) + 1]$
$= max_{1 \leq m \leq k}[m - 1 + k + 1 - m - 1 + 1]$
$= max_{1 \leq m \leq k}[k] = k$. $(k \geq 1)$
So, it still holds.
Conclusion: $g(n) = n - 1$ for all $n \geq 1$.

11. (a) What is the reflexive-symmetric-transitive closure of the relation

$$R = \{(1,2), (1,3), (2,4), (5,6)\}$$

defined on the set $A = \{1, 2, 3, 4, 5, 6\}$.
**Solution:** Reflexive: Adding (1,1),(2,2),(3,3)(4,4),(5,5),(6,6).
Symmetric: Adding (2,1),(3,1),(4,2),(6,5).
Transitive: Adding (1,4),(4,1),(2,3),(3,2),(3,4),(4,3).
The final result is the original pairs in R and all the added ones.

(b) How many different binary relations on a set A of cardinality $n$ are both symmetric and reflexive?

**Solution:** $|A| = n$. To satisfy the reflexive requirement, we should add all pairs of $(a_i, a_i)$. Then, we can freely select the pairs of the form that $(a_i, a_j), i < j$(and the opposite one$(a_j, a_i)$), to satisfy the "symmetric" requirement. There are totally $n(n-1)/2$ such $(a_i, a_j), i < j$, and each of them can be in or not in the relation. So the number of possible choices is $2^{n(n-1)/2}$.
So, the number of the possible relations is:
$N = 2^{n(n-1)/2}$

12. Consider 6 letter words (not necessarily meaningful) over an alphabet of 26 letters.

    (a) How many different 6 letter words are there?
    **Solution:** $26^6$.

    (b) How many different 6 letter words are there with at least one repeated letter?
    **Solution:** If there is no repeated letter, the number of the possible choices is $P(26, 6)$. So, the number of the words with at least one repeated letter is $26^6 - P(26, 6)$, where P(26,6) is the computation of permutation.

    (c) Consider the relation $R$ on 6 letter words, defined by $w_1 R w_2$ if and only if $w_1$ is the reverse of $w_2$. For example, with $w_1 =$ aabcde and $w_2 =$ edcbaa, we have $w_1 R w_2$. Is $R$ an equivalence relation? If not, why not?
    **Solution:** R is not an equivalence relation. Because it doesn't satisfy the reflexive requirement. For example, $w_1 = aabcde$,$w_1 R w_1$ is not correct.

    (d) Consider the relation $R$ on 6 letter words, defined by $w_1 R w_2$ if and only if $w_1$ is a permutation of $w_2$. For example, with $w_1 =$ aabcde and $w_2 =$ ecaadb, $w_1 R w_2$.

       i. Is this an equivalence relation? If not, why not?
       **Solution:** Yes. It satisfies all the requirements: transitive, reflexive and symmetric.

       ii. If so, how many words are in the equivalence class [aaabbc]?
       **Solution:** First consider the "c", and it has 6 possible positions. Then for the two "b"s, the number of the possible choices is $C(5, 2)$. So, there are $6C(5, 2)$ words in the class.

13. Let $G$ be a complete (i.e., every edge is present), simple, undirected graph on $n$ nodes. Color each of the edges independently either red or blue equally likely (so the probability that a particular edge is colored red is 1/2). Let $S$ be a particular subset of $k$ nodes in $G$. What is the probability that all the edges that have both endpoints in $S$ are colored red? (We say such a subset $S$ is *red and monochromatic.*)
    **Solution:** If a set S is red and monochromatic, we denote $S = RM$.
    For a particular subset S containing k nodes, the number of the edges contained in S is $k(k-1)/2$. So, the number of possible coloring methods is $2^{k(k-1)/2}$. Then, the possibility that $S = RM$ is $1/2^{k(k-1)/2}$.

14. Exact same setup as previous question: What is the expected number of subsets of $k$ vertices (out of the $n$ vertices total) that are red and monochromatic? (Hint: use linearity of expectation.)
    **Solution:** Given a particular S containing k nodes, from the above problem, we know

the probability that S is red and monochromatic is $1/2^{k(k-1)/2}$. Totally, there are $C(n,k)$ possible such S, so the expected number is:
E(N)=$C(n,k)2^{-k(k-1)/2}$.

15. Let $A$ be the set of all undirected, simple graphs on $n$ nodes. Define a relation $R$ on $A$ as follows: Two graphs $G$ and $G'$ in $A$ are related by $R$ if there is a bijection $f$ from the vertices of $G$ to the vertices of $G'$ such that $(u,v)$ is an edge in $G$ if and only if $(f(u), f(v))$ is an edge in $G'$. *True or false:* $R$ is an equivalence relation.

    **Solution:** True. In fact the relation partitions the set of all possible undirected, simple graphs into equivalence classes of isomorphic graphs. The relation is reflexive since, for a given graph $G$ the identity function is a bijection $f$ that satisfies the definition. Further, if $f$ is a bijection for $(G, G')$, then note that inverse function $f^{-1}$ is a bijection for $(G', G)$. Therefore the relation is symmetric. Further if $(G_1, G_2) \in R$ with a corresponding bijection $f$, and $(G_2, G_3) \in R$ with a correspoding bijection $g$, then $g \circ f$ is a bijection from $G_1$ to $G_3$ satisfying the above property. Therefore $(G_1, G_3) \in R$ and $R$ is a transitive relation.

16. Prove by induction that if $n$ is an odd, positive integer, $n^2 - 1$ is divisible by 4. (Write out your solution, as I did in class, in 5 steps labelled as follows: base case, inductive hypothesis, to prove, inductive step, and conclusion.)

    **Solution:** Base case: $n = 1$: $n^2 - 1 = 0$ is divisible by 4.
    Inductive hypothesis: if $n$ is odd positive integer, $n^2 - 1$ is divisible by 4.
    To prove: $(n+2)^2 - 1$ is divisible by 4.
    Inductive step:
    $(n+2)^2 - 1 = n^2 + 4n + 4 - 1 = (n^2 - 1) + 4(n+1)$.
    By inductive hypothesis, there exists $k$ s.t. $n^2 - 1 = 4k$.
    Thus, $(n+2)^2 - 1 = 4k + 4(n+1) = 4(n+k+1)$.
    Then, by definition of divisibility,$(n+2)^2 - 1$ is divisible by 4.
    Conclusion: by the principle of mathematical induction, for all $n$, if $n$ is odd, positive integer then $n^2 - 1$ is divisible by 4.

17.
    - In the following sentence, fill in both blanks with the smallest integer such that your answer is guaranteed to be correct for all simple planar graphs.

        Recall that in a simple planar graph with $n$ nodes, the number of edges is at most $3n - 6$. This implies that the sum of the degrees of all the nodes is at most ............ $6n - 12$. Consequently, there is always a node of degree at most ............ $\lfloor (6n-12)/n \rfloor \leq 5$.

    - Use your answer to the previous question to prove by induction that every planar graph is 6 colorable. (Write out your solution, as I did in class, in 5 steps labelled as follows: base case, inductive hypothesis, to prove, inductive step, and conclusion.)

        **Solution:** Let $G_n = (V, E)$ be a simple planar graph, where $|V| = n$.
        Base case: $n = 1$: $G_1$ is 1 colorable $\Rightarrow$ $G_1$ is 6 colorable.
        Inductive hypothesis: $G_n$ is 6 colorable.

To prove:$G_{n+1}$ is 6 colorable.
Inductive step:
By the previous question, there is a node $v \in V$ s.t. $d(v) = k \leq 5$. Let $G' = (V', E')$, where $V = V \setminus \{v\}$, $E = \{(u, w) \in E | u \neq v \wedge w \neq v\}$ ($G'$ is obtained from $G$ by removing $v$ and all its adjacent edges). Then $G'$ is a planar graph with $n$ nodes. By inductive hypothesis, $G'$ is 6 colorable. Let $c_1, \ldots, c_k$ be the colors of the nodes adjacent to $v$. Since $k \leq 5 < 6$ we can color $v$ in the one of the remaining colors.
Conclusion: every planar graph is 6 colorable.

18. Show that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

    **Solution:** Let $G = (V, E)$ be a simple graph, $v \in V$ a node of odd degree. Perform a walk in the graph $G$ starting at node $v$,where at each step an edge is chosen from previously unvisited edges, until the some other node of odd degree is reached or a step cannot be performed anymore.

    - if the other node of odd degree is reached, we are done.
    - If we cannot perform a step anymore, we are not at the first node - otherwise an even number of adjacent edges is visited, which means that the degree of $v$ is even, contradicting the choice of $v$. Thus, each node in the path, except for first and last, has even number of adjacent edges visited. The first and last have odd number of adjacent edges visited. Since we cannot perform a step, every edge adjacent to the last node has been visited, which implies that the degree of a last node is odd.

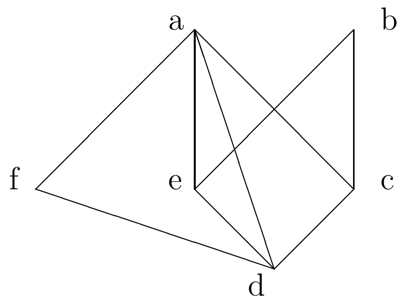
    In both cases, there exits a path from $v$ to some other node of odd degree. Q.E.D.

19. For which values of $m$ and $n$ does the complete bipartite graph $K_{m,n}$ have an

    - Euler circuit
      **Solution:** For even $m$ and $n$
    - Euler path.
      **Solution:** For $(m = 2, n\ is\ odd)$, $(m\ is\ odd, n = 2)$, and $(m = 1, n = 1)$.

20. Questions like the following where a picture of some graph is given:

    (a) Is the following graph connected? Yes
    (b) Draw an Euler tour for the following graph. $e \to a \to f \to d \to e \to b \to c \to d \to a \to c$
    (c) Is the following graph planar? If so, how many faces does it have? Yes, 5
    (d) Can the following graph be 3 colored? Yes


a
b
f
e
c
d