CSE 321
Practice Problems from Old Finals

Instructions:

- Feel free NOT to multiply out binomial coefficients, factorials, etc, and feel free to leave answers in the form of a sum.
- No calculators, books or notes are allowed.

1. True or False:

   - $p \to q$ is logically equivalent to $q \to p$.
   - $((p \to q) \wedge \neg p) \to \neg q$ is a tautology.
   - $((\forall x[P(x) \to Q(x)]) \wedge P(y)) \to Q(y)$ is a tautology.
   - There is a one-to-one function from A to B if and only if there exists an onto function from B to A.
   - To prove by contradiction that $p \to q$, one must show that p is false.
   - $Pr(A \bigcup B) \leq Pr(A) + Pr(B)$.
   - For any event A in a probability space $0 \leq Pr(A) \leq 1$.
   - For any events A and B in a probability space $Pr(A \mid B) = Pr(A)$.
   - An undirected graph has an even number of vertices of odd degree.

2. On the next set of questions, fill in the blanks.

   - If a set A is contained in a set B, then $A \cup B =$ ....
   - If a set A is contained in a set B, then $A \cap B =$ ....
   - The number of subsets of an n element set is ....
   - The number of ways of choosing an unordered subset of size k out of a set of size r is ....
   - The coefficient of x^{10} in the polynomial $(5x+1)^{100}$ is ....

- The number of different binary relations from a set A of size n to a set B of size m is ....
- The number of different reflexive binary relations on a set A of size n is ....
- The number of different undirected graphs (no self loops and no parallel edges) on n vertices is ....
- What is the coefficient of x^7 in $(10x+2)^{21}$?
- What is the probability of getting exactly 12 heads if a biased coin with probability 4/5 of coming up heads is tossed 25 times (independently)?

3.
- Every day, starting on day 0, one vampire arrives in Seattle from Transylvania and, starting on the day after its arrival, bites one Seattlite every day. People bitten become vampires themselves and live forever. New vampires also bite one person each day starting the next day after they were bitten. Let V_n be the number of vampires in Seattle on day n. So, for example, $V_0 = 1$, $V_1 = 3$ (one that arrived from Transylvania on day 0, one that he bit on day 1, and another one that arrived from Transylvania on day 1), $V_2 = 7$ and so on. Write a recurrence relation for V_n that is valid for any $n \geq 2$.
- Prove by induction on n that $V_n \leq 3^n$.

4. (25 points) Consider an exam consisting of 25 True/False questions. Suppose that a student has probability 1/2 of getting the answer to a particular question right, independently for all questions.
   (a) In how many different ways can the student answer the questions?
   (b) What is the probability that the student answers the second question correctly given that the student answers the first question correctly?
   (c) What is the probability that the student answers the first two questions correctly given that the student answers *at least* one of the first two questions correctly?
   (d) What is the expected number of answers the student gets right? Briefly explain your answer.
   (e) What is the expected number of points the student gets on the exam if the student gets 2 points for each question answered correctly and gets 1 point taken away (or equivalently -1 point) for each question answered incorrectly?

5.
- How many permutations of the letters {a,b,c,d,e,f,g,h} are there?
- How many permutations of {a,b,c,d,e,f,g,h} are there that don't contain the letters "bad" (appearing consecutively)?
- How many permutations of {a,b,c,d,e,f,g,h} are there that don't contain either the letters "bad" appearing consecutively or the letters "fech" appearing consecutively?
- How many words of length 10 can be constructed using the letters {a,b,c,d,e,f,g,h} that contain exactly 3 a's? (They don't have to have any English meaning.)

6. Suppose a biased coin with probability $3/4$ of coming up heads is tossed independently 100 times.

   - What is the conditional probability that the first 50 tosses are heads given that the total number of heads is 50?
   - What is the expected number of heads?
   - Suppose that you are paid \$50 if the number of heads in the first **two** tosses is even and \$100 if the number of heads in these first two tosses is odd. What is your expected return?

7. Use the Euclidean algorithm to find gcd (486, 446).

8. A lake contains n trout. 100 of them are caught, tagged and returned to the lake. Later another set of 100 trout are caught, selected independently from the first 100.

   - Write an expression (in n) for the probability that of the second 100 trout caught, there are exactly 7 tagged ones.
   - Now consider selecting the second 100 trout with replacement. That is, you repeat 100 times the following steps: select at random a trout in the lake, check if the trout is tagged and return it to the lake before selecting a new trout. What is the probability that exactly 7 of the selected trout are tagged?

9. Prove that any undirected, connected graph with n vertices and no cycles (i.e., a tree) has exactly $n-1$ edges. (You should assume that the graph has no self loops and no parallel edges. A cycle is a sequence of 2 or more distinct edges that start and end at the same vertex.)

10. Suppose that for all $n \geq 1$

$$g(n+1) = max_{1 \leq k \leq n}[g(k) + g(n+1-k) + 1]$$

and that $g(1) = 0$. Prove by induction that $g(n) = n - 1$ for all $n \geq 1$.

11. (a) What is the reflexive-symmetric-transitive closure of the relation

$$R = \{(1,2), (1,3), (2,4), (5,6)\}$$

defined on the set $A = \{1,2,3,4,5,6\}$.

(b) How many different binary relations on a set A of cardinality n are both symmetric and reflexive?

12. Consider 6 letter words (not necessarily meaningful) over an alphabet of 26 letters.

(a) How many different 6 letter words are there?

(b) How many different 6 letter words are there with at least one repeated letter?

(c) Consider the relation R on 6 letter words, defined by $w_1 R w_2$ if and only if w_1 is the reverse of w_2. For example, with $w_1 =$ aabcde and $w_2 =$ edcbaa, we have $w_1 R w_2$. Is R an equivalence relation? If not, why not?

(d) Consider the relation R on 6 letter words, defined by $w_1 R w_2$ if and only if w_1 is a permutation of w_2. For example, with $w_1 =$ aabcde and $w_2 =$ ecaadb, $w_1 R w_2$.

   i. Is this an equivalence relation? If not, why not?

   ii. If so, how many words are in the equivalence class [aaabbc]?

13. Let G be a complete (i.e., every edge is present), simple, undirected graph on n nodes. Color each of the edges independently either red or blue equally likely (so the probability that a particular edge is colored red is 1/2). Let S be a particular subset of k nodes in G. What is the probability that all the edges that have both endpoints in S are colored red? (We say such a subset S is *red and monochromatic.*)

14. Exact same setup as previous question: What is the expected number of subsets of k vertices (out of the n vertices total) that are red and monochromatic? (Hint: use linearity of expectation.)

15. Let A be the set of all undirected, simple graphs on n nodes. Define a relation R on A as follows: Two graphs G and G' in A are related by R if there is a bijection f from the vertices of G to the vertices of G' such that (u, v) is an edge in G if and only if $(f(u), f(v))$ is an edge in G'. *True or false:* R is an equivalence relation.

16. Prove by induction that if n is an odd, positive integer, $n^2 - 1$ is divisible by 4. (Write out your solution, as I did in class, in 5 steps labelled as follows: base case, inductive hypothesis, to prove, inductive step, and conclusion.)

17.
    - In the following sentence, fill in both blanks with the smallest integer such that your answer is guaranteed to be correct for all simple planar graphs.

      Recall that in a simple planar graph with n nodes, the number of edges is at most $3n - 6$. This implies that the sum of the degrees of all the nodes is at most .... Consequently, there is always a node of degree at most ....

    - Use your answer to the previous question to prove by induction that every planar graph is 6 colorable. (Write out your solution, as I did in class, in 5 steps labelled as follows: base case, inductive hypothesis, to prove, inductive step, and conclusion.)

18. Show that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

19. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

    - Euler circuit
    - Euler path.

20. Questions like the following where a picture of some graph is given:

    (a) Is the following graph connected?
    (b) Draw an Euler tour for the following graph.
    (c) Is the following graph planar? If so, how many faces does it have?
    (d) Can the following graph be 3 colored?