CSE 321 Winter 2007 Final Exam Solutions

1. (8 points) Sam Slacker usually oversleeps and misses his bus. Sometimes luck is on his side and the bus is running late, but even then he might still miss the bus. If the bus is running late, he catches it 70% of the time. If the bus is running on time, he catches it only 20% of the time. Metro Bus runs a fairly tight ship and busses run late only about 30% of the time. What is the probability that the bus was running late given that Sam managed to catch the bus?

   Solution: Let us define some variables for the events that we are interested in. L will represent the event that the bus is late. C will represent the event that Sam catches the bus. Then we are given the following probabilities:

   - $Pr[C|L] = 0.7$
   - $Pr[C|\neg L] = 0.2$
   - $Pr[L] = 0.3$

   We can of course also infer that $Pr[\neg L] = 1 - Pr[L] = 1 - 0.3 = 0.7$.

   Now, we want to compute the value $Pr[L|C]$. We do this using Bayes' Rule.

$$\begin{aligned}
Pr[L|C] &= \frac{Pr[C|L] \cdot Pr[L]}{Pr[C|L] \cdot Pr[L] + Pr[C|\neg L] \cdot Pr[\neg L]} \\
&= \frac{0.7 \cdot 0.3}{0.7 \cdot 0.3 + 0.2 \cdot 0.7} \\
&= \frac{0.21}{0.35} \\
&= \frac{21}{35}
\end{aligned}$$

   So the probability of the bus running late given that Sam caught it is $\frac{21}{35}$.

2. (8 points) Show that for an any positive integer n, $4n + 3$ and $5n + 4$ are relatively prime. (Hint: Use Euclid's algorithm. Do not use induction.)

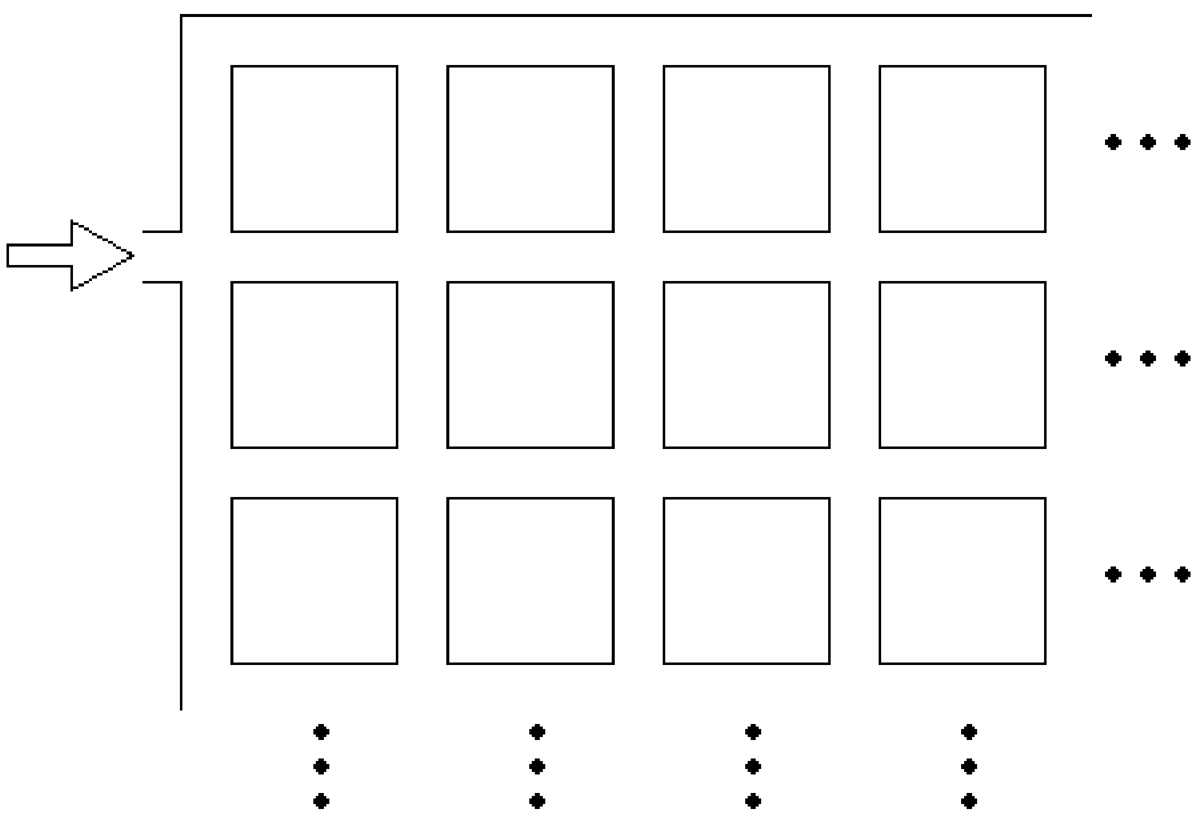
   Solution: We use Euclid's algorithm in the standard way.

$$\begin{aligned}
5n + 4 &= 1 \cdot (4n + 3) + (n + 1) \\
4n + 3 &= 3 \cdot (n + 1) + n \\
n + 1 &= 1 \cdot n + 1
\end{aligned}$$

   Thus 1 is the greatest common denominator of $4n+3$ and $5n+4$, which is the definition of being relatively prime.

3. (8 points) The gated town of Squaresville is made up of 10 by 10 square blocks (see partial picture below). Sam Slacker is the new mailman for Squaresville. Everyday, he enters Squaresville at their only entrance (see arrow below) and delivers mail to all the residents.

   Being a slacker, Sam wants to do as little work as possible; he is looking for a way to reach every house without having to double back. Assume that once Sam passes a house (on either side of the street), then he can deliver mail to that house. Can Sam deliver mail to everyone without doubling back or retracing any of his steps? Why or why not?



   Solution: We can model Squaresville perfectly as an undirected graph, in which the intersections of the streets are vertices, and the streets themselves are edges. If Sam delivers his mail without doubling back or retracing his steps, such a route must be an Euler path, since it requires crossing every edge exactly once. However, we know that an undirected graph with more than two vertices of odd degree cannot have an Euler path. In the map of Squaresville, all non-corner vertices on the edge of the map have odd degree, and hence no Euler path is possible. Sam cannot deliver his mail without doubling back or retracing his steps. If we require Sam to return back to the gate so he can go home, we can use a similar argument and show that there is no Euler circuit, because there are vertices with odd degrees.

4. (8 points)

   (a) "Mega Millions is the multi-state lottery game that has the biggest jackpot around. ... Just select five numbers from a field of 56, and any one number from a field of 46." The five numbers are unique and their order does not matter. The sixth number is not necessarily different from any of the first five numbers. How many ways are there to select six numbers to play Mega Millions?

   Solution: There are $\binom{56}{5}$ ways to choose 5 unique numbers from a field of 56. There are $\binom{46}{1} = 46$ ways to choose 1 number from a field of 46. Our choices from the first and second sets of numbers are independent, so by the product rule, we can take the product of the number of ways to choose each. Thus there are $46 \cdot \binom{56}{5}$ ways to play Mega Millions.

   (b) If the rules were changed so that the five numbers did not have to be unique, how many ways would there be to select six numbers to play Mega Millions?

   Solution: We are choosing five numbers, each of which can be from one of 56 types, but order doesn't matter. This is exactly stars and bars, because we can use each number as many times as we like. In this case there are 55 bars and 5 stars. Thus there are $\binom{55+5}{5}$ ways to pick the five numbers. There are still 46 ways to choose the sixth number, so there are $46 \cdot \binom{60}{5}$ ways to play Mega Millions in this variant.

5. (8 points) Prove that $(q \wedge (p \to \neg q)) \to \neg p$ is a tautology without using a truth table.

Solution: We use a sequence of logical equivalences in the standard way.

$$
\begin{aligned}
(q \wedge (p \to \neg q)) \to \neg p &\equiv (q \wedge (\neg p \vee \neg q)) \to \neg p && \text{(Implication Law)}\\
&\equiv ((q \wedge \neg p) \vee (q \wedge \neg q)) \to \neg p && \text{(Distributive Law)}\\
&\equiv ((q \wedge \neg p) \vee \mathbf{F}) \to \neg p && \text{(Negation Law)}\\
&\equiv (q \wedge \neg p) \to \neg p && \text{(Identity Law)}\\
&\equiv \neg(q \wedge \neg p) \vee \neg p && \text{(Implication Law)}\\
&\equiv (\neg q \vee \neg\neg p) \vee \neg p && \text{(De Morgan's Law)}\\
&\equiv (\neg q \vee p) \vee \neg p && \text{(Double Negation Law)}\\
&\equiv \neg q \vee (p \vee \neg p) && \text{(Associative Law)}\\
&\equiv \neg q \vee \mathbf{T} && \text{(Negation Law)}\\
&\equiv \mathbf{T} && \text{(Domination Law)}
\end{aligned}
$$

Since the statement is equivalent to True, it is a tautology, and we are done.

6. (8 points) A palindrome is a string whose reversal is identical to the string. For strings over an alphabet of 26 letters (*e.g.*, the Roman alphabet), "kayak" and "qrexxerq" are examples of palindromes. Notice that the string does not necessarily have to be an English word. How many such strings of length n are palindromes? (Hint: Do not try to come up with a single formula for all n.)

   Solution: Let us first consider the case in which n is even. Then a palindrome can have any character in its first position, any in its second, etc., all the way until its $\frac{n}{2}$'th character. After that, each character is fixed, because it is forced to equal the mirrored character in the first half. Thus we have $\frac{n}{2}$ independent choices from the alphabet, with replacement, where order matters. There are $26^{n/2}$ palindromes of even length.

   If n is odd, then the method is the same, except that the middle character does not have a mirrored copy. It can be chosen freely, however. Thus the total number of 'free' characters is $\frac{n-1}{2} + 1 = \frac{n+1}{2}$. Hence there are $26^{(n+1)/2}$ palindromes of odd length.

7. (8 points)

   (a) Consider the following relation over the set of sets.

   $$R = \{(A, B) \mid A \cap B \neq \emptyset\}$$

   Is R an equivalence relation? Why or why not?

   Solution: No. An equivalence relation is reflexive, symmetric, and transitive. Of the three, R is not transitive.

   Let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{2\}$. Note that A is related to B, and B is related to C, but A is not related to C.

   (b) If so, let $A = \{1, 2, 3\}$. What is another set in the equivalence class of A? If not, leave this space blank.

8. (12 points)

(a) A pair of cubical (standard) dice are rolled together. I'll pay you \$4 if the sum of the numbers are 10 or higher, otherwise you'll pay me \$1. What is your expected payoff?

Solution:
Let $S = \{(r_1, r_2) \mid r_1 \text{ is the number of the first die and } r_2 \text{ is the number of the second die}\}$
Let X be the random variable:

$$X((r_1, r_2)) = \begin{cases} 4 & r_1 + r_2 \geq 10 \\ -1 & r_1 + r_2 < 10 \end{cases}$$

The expected value of X is:

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

The probability $p(X = 4)$ is the count of the number of times that the sum is greater than or equal to 10 over the total number of rolls, which we know to be 36 (there are 6 possibilities per die and there are two dice).

Let E_i be the event that the sum is equal to i. Let E be the event that the sum is greater than or equal to 10. Since the events E_i are disjoint, $|E| = |E_{10}| + |E_{11}| + |E_{12}|$.

$$\begin{aligned} E_{10} &= \{(4,6), (5,5), (6,4)\} \\ E_{11} &= \{(5,6), (6,5)\} \\ E_{12} &= \{(6,6)\} \end{aligned}$$

Thus $|E| = |E_{10}| + |E_{11}| + |E_{12}| = 3 + 2 + 1 = 6$. That means that the 30 other possibilities comprise the case when the sum is less than 10.

The probability that $X = 4$ is $\frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$. The probability that $X = -1$ is the remaining probability, $\frac{5}{6}$.

Your expected payoff is then

$$\begin{aligned} E(X) &= p(X = 4) \cdot 4 + p(X = -1) \cdot (-1) \\ &= \frac{1}{6} \cdot 4 + \frac{5}{6} \cdot (-1) \\ &= \frac{4}{6} - \frac{5}{6} \\ &= -\frac{1}{6} \end{aligned}$$

(b) An *octahedral* die has eight (8) faces that are numbered 1 through 8. A *dodecahedral* die has twelve (12) faces that are numbered 1 through 12. What is the expected value of the sum of the numbers that come up when a fair octahedral die and a fair dodecahedral die are rolled together?

Solution: Let X_1 be the value of the octahedral die and X_2 be the value of the dodecahedral die. Let X be the sum of the numbers rolled, then $X = X_1 + X_2$. By the linearity of expectation, $E(X) = E(X_1) + E(X_2)$.

$$\begin{aligned} E(X_1) &= \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 4 + \frac{1}{8}\cdot 5 + \frac{1}{8}\cdot 6 + \frac{1}{8}\cdot 7 + \frac{1}{8}\cdot 8 \\ &= \frac{1}{8}\cdot (1+2+3+4+5+6+7+8) \\ &= \frac{1}{8}\cdot 36 \\ &= 4.5 \end{aligned}$$

Similarly,

$$\begin{aligned} E(X_2) &= \frac{1}{12}\cdot (1+2+3+4+5+6+7+8+9+10+11+12) \\ &= \frac{1}{12}\cdot 78 \\ &= 6.5 \end{aligned}$$

Thus, $E(X) = 4.5 + 6.5 = 11$.

9. (8 points) Prove the statement: "For any integers a, b, and c, if there exists a positive integer m such that $b \not\equiv c \pmod{m}$, then either $a \neq b$ or $a \neq c$."

Solution: We are going to prove this via the contrapositive. Assume that $a = b$ and $a = c$. Then $b = c$. Since $b = c$, there cannot exist any m such that $b \not\equiv c \pmod{m}$. Every positive integer divides 0, thus every candidate m will divide $b - c = 0$.

10. (8 points)

   (a) Consider a relation R over the set $S = \{a, b, c, d, e\}$. How many such relations are there? Explain. Try not to use more than 50 words.

   Solution: A relation over a set S is a subset of the Cartesian product $S \times S$. R is a subset of

$$S \times S = \{(a, b) \mid a \in S \wedge b \in S\}$$

   There are 25 such tuples, since there are 5 elements in S. A subset is like a bit string where a 0 indicates that the tuple is not present, while a 1 indicates that it is. Since there are 25 such "bits" to choose from, there are 2^{25} such relations.

   (b) Suppose you chose one such relation uniformly at random. What is the probability that the size of the relation is at least 3?

   Solution: We first count how many relations have a size of less than 3. There are $\binom{25}{0}$ relations of size 0, $\binom{25}{1}$ relations of size 1, and $\binom{25}{2}$ relations of size 2.

   The probability that the relation has size less than 3 is

$$\frac{\binom{25}{0} + \binom{25}{1} + \binom{25}{2}}{2^{25}}$$

   Thus, the probability that the relation has size at least three is

$$1 - \frac{\binom{25}{0} + \binom{25}{1} + \binom{25}{2}}{2^{25}}$$

11. (8 points) Prove that at a party where there are at least two people, there are two people who know the same *number* of other people there. Assume that everyone knows at least one other person.

   Solution: Suppose there are n people at the party. Every person can know up to $n-1$ other people. By the pigeonhole principle, at least two people know the same number of other people.

12. (8 points) Prove that for every positive integer n,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Solution:
Let $P(n)$ be the proposition "$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$"

Base Case: We prove that $P(1)$ holds.

$$\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{1\cdot 3}=\frac{1}{3}=\frac{1}{2\cdot 1+1}$$

Thus, $P(1)$ holds.

Inductive Hypothesis: Assume that $P(n)$ holds.
Inductive Step: We wish to prove that $P(n+1)$ holds, given the inductive hypothesis.

$$\begin{aligned}
\frac{1}{1\cdot 3} &+ \frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2(n+1)-1)(2(n+1)+1)} \\
&= \frac{n}{2n+1}+\frac{1}{(2(n+1)-1)(2(n+1)+1)} \\
&= \frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)} \\
&= \frac{n\cdot(2n+3)}{(2n+1)(2n+3)}+\frac{1}{(2n+1)(2n+3)} \\
&= \frac{n\cdot(2n+3)+1}{(2n+1)(2n+3)} \\
&= \frac{2n^2+3n+1}{(2n+1)(2n+3)} \\
&= \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} \\
&= \frac{n+1}{2n+3} \\
&= \frac{n+1}{2(n+1)+1}
\end{aligned}$$

where the first equality holds from the inductive hypothesis.
This proves that $P(n+1)$ holds. Thus, by induction, $P(n)$ holds $\forall n\geq 1$.