Sums and Recurrences
Notes for CSE 321 – Winter 2010

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Sums

$$\sum_{i=1,n} f(i) \quad = \quad f(1) + f(2) + \ldots + f(n)$$

Sometimes we want to start at 0:

$$\sum_{i=0,n} f(i) \quad = \quad f(0) + f(1) + f(2) + \ldots + f(n)$$

Telescoping

Find a function g such that $f(x) = g(x) - g(x-1)$ then write:

$$\begin{aligned} f(1) + f(2) + \ldots + f(n) &= (g(1) - g(0)) + (g(2) - g(1)) + \ldots (g(n) - g(n-1)) \\ &= g(n) - g(0) \end{aligned}$$

For example, we use: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ to compute:

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} &= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

Rule of Thumb: There are only three ways to compute a sum *exactly*: (1) telescoping, or (2) someone tells you what the sum is, and you prove it by induction (and this is, ultimately, the same as (1)), or (3) you derive it from another sum, which you computed using (1) or (2).

Approximating Sums with Integrals

Sometimes we don't need to compute the sum exactly, but we want to approximate it: then we use integrals. Suppose $f(x)$ is strictly increasing, then:

$$\int_{i-1}^{i} f(x)dx < f(i) < \int_{i}^{i+1} f(x)dx.$$

(Draw a graph to convince yourself why this is the case.) Hence:

$$\int_{0}^{n} f(x)dx < f(1) + f(2) + \ldots + f(n) < \int_{1}^{n+1} f(x)dx$$

For example, we want to approximate the sum:

$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$$

Here $f(x) = \sqrt{x}$, which is strictly increasing. Since $\int \sqrt{x}dx = \frac{2 \cdot x^{3/2}}{3}$ we have

$$\frac{2 \cdot \sqrt{n^3}}{3} < \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \frac{2 \cdot \sqrt{(n+1)^3}}{3} - \frac{2}{3}$$

Suppose f is strictly decreasing: then replace $<$ with $>$ in all inequalities aboved.

For example, we want to approximate the sum:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Here $f(x) = \frac{1}{x}$, which is strictly decreasing. Since $\int \frac{dx}{x} = ln(x)$, we have:

$$1 + (\int_{1}^{n} \frac{dx}{x}) = 1 + ln(n) > \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_{1}^{n+1} \frac{dx}{x} = ln(n+1)$$

What happened on the left ? We couldn't compute $\int_0^n f(x)dx$, because $ln(0)$ is undefined. Instead we wrote $\sum_{i=1,n} 1/i$ as $1 + \sum_{i=2,n} 1/i$; the upper bound for this is $1 + \int_1^n f(x)dx$.

The integral method is very powerful. The textbook shows a weaker bound for the harmonic series: $H_n = \sum_{i=1,n} 1/i > 1 + \log_2(n)/2$. The lower bound $ln(n+1)$ that we obtained here is better.

Rule of thum If you don't need to compute the sum exactly, but only to approximate it, then the integral method is the quickest method, and the best method. There is only one reason to use another method than the integral: when you can't compute $\int f(x)dx$. On the other hand, we can't use integrals when we need to compute the sum exactly.

Sums of Polynomials

Compute these sums exactly:

$$\begin{array}{l} 1+2+3+\ldots+n \\ 1^2+2^2+3^2+\ldots+n^2 \\ 1^3+2^3+3^3+\ldots+n^3 \\ \ldots \\ 1^k+2^k+3^k+\ldots+n^k \end{array}$$

More generally, let $P(x)$ be a polynomial in x. Compute this sum exactly:

$$P(1) + P(2) + \ldots + P(n)$$

Consider the following special polynomial:

$$(x)_k = x \cdot (x-1) \cdot (x-2) \cdots (x-k+1)$$

The sum of $(x)_k$ is:

$$\begin{aligned} S_k(n) &= (1)_k + (2)_k + (3)_k + \ldots + (n)_k \\ &= 1 \cdot 2 \cdots k + 2 \cdot 2 \cdots (k+1) + \ldots + (n-k+1) \cdots n \end{aligned}$$

We compute $S_k(n)$ using telescoping:

$$\begin{aligned}
(x+1)_{k+1} - (x)_{k+1} &= (x+1)\cdot x\cdot(x-1)\cdot(x-2)\cdots(x-k+1) - \\
&\quad -x\cdot(x-1)\cdot(x-2)\cdots(x-k) \\
&= [(x+1)-(x-k)]x\cdot(x-1)\cdot(x-2)\cdots(x-k+1) \\
&= (k+1)\cdot(x)_k
\end{aligned}$$

The sum becomes:

$$\begin{aligned}
S_k(n) &= (1)_k + (2)_k + (3)_k + \ldots + (n)_k \\
&= \frac{1}{k+1}\left[(2)_{k+1} - (1)_{k+1} + (3)_{k+1} - (2)_{k+1} + \ldots (n+1)_{k+1} - (n)_{k+1}\right] \\
&= \frac{(n+1)_{k+1}}{k+1}
\end{aligned}$$

So we have computed $\sum_{i=1,n}(i)_k$. Let's go back to computing $\sum_{i=1,n} i^k$. For that we express x^k as a linear combination of $(x)_1, (x)_2, \ldots, (x)_k$. The principled way to do this is through *Stirling numbers of the second kind*, but if this sounds scary, here is a simpler way. Expand $(x)_1, (x)_2, (x)_3, \ldots$:

$$\begin{aligned}
(x)_1 &= x \\
(x)_2 = x(x-1) &= x^2 - x \\
(x)_3 = x(x-1)(x-2) &= x^3 - 3x^2 + 2x \\
(x)_4 = x(x- =)(x-2)(x-3) &= \ldots \\
\ldots
\end{aligned}$$

"Solve" for $x, x^2, x^3, \ldots$:

$$\begin{aligned}
x &= (x)_1 \\
x^2 &= (x)_2 + (x)_1 \\
x^3 &= (x)_3 - 3(x)_2 - (x)_1 \\
x^4 &= (x)_4 + \ldots \\
\ldots
\end{aligned}$$

Now we can compute any sum $\sum_{i=1,n} i^k$. For example, take $\sum_{i=1,n} i^3$:

$$\begin{aligned}
1^3 + 2^3 + \ldots + n^3 &= \sum i^3 = \sum (i)_3 - 3\sum (i)_2 - \sum (i)_1 \\
&= \frac{(n+1)_4}{4} - 3\frac{(n+1)_3}{3} - \frac{(n+1)_2}{2} \\
&= \frac{(n+1)n(n-1)(n-2)}{4} - 3\frac{(n+1)n(n-1)}{3} - \frac{(n+1)n}{2} \\
&= \frac{n(n+1)}{4}\left((n-1)(n-2) - 4(n-1) - 2\right) \\
&= \frac{n(n+1)}{4}\left(n^2 - 3n + 2 - 4n + 4 - 2\right) \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

We can compute now $\sum P(i)$ for any polynomial $P(x)$. For example: "what is the sum of the first n odd numbers" ?

$$\begin{aligned}
1 + 3 + 5 + \ldots + (2n-1) &= \sum_{i=1,n} (2i-1) \\
&= 2\sum_{i=1,n} i - \sum_{i=1,n} 1 = 2\frac{n(n+1)}{2} - n = n^2
\end{aligned}$$

Answer: "the sum of the first n odd numbers is n^2".

Rule of Thumb: To compute sums of any polynomial, first memorize the three standard sums:

$$\begin{aligned}
\sum_{i=1,n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1,n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1,n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

If you have to go beyond $k = 3$, then you'll need to derive it from $\sum_i (i)_k$.

Power Sums

Prove the following identity by induction on n:

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \tag{1}$$

When $|x| < 1$, then $\lim_{n \to \infty} x^{n+1} = 0$. Hence, we obtain the infinite series:

$$1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1 - x} \quad \text{when } |x| < 1 \tag{2}$$

Alice has a pizza. On the first day she eats half of the pizza; on the second day she eats half of what is left; on the third day she eats half of what is left, and so on. On each day she eats half of what is left. In how many days will she finish the pizza ?

Answer: never ! This is because on day n she eats $1/2^n$ of the pizza, hence, given an infinite amount of time she will eat:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} + \ldots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots\right) = \frac{1}{2}\frac{1}{1 - \frac{1}{2}} = 1$$

Bob watched Alice eating her pizza and tried to eat even more: on day n, Bob eats n-times the amount of pizza that Alice ate that day. Thus, on day 1 Bob ate the same amount as Alice ($1/2$ a pizza); on day 2 he ate twice what Alice ate ($2 * 1/2^2$ of a pizza); on day 3 he ate three times Alice's portion ($3 * 1/2^3$), etc. How much more pizza does Bob eat than Alice ? Here's what Bob eats:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} + \ldots = \frac{1}{2}\left(1 + \frac{2}{2} + \frac{3}{2^2} + \ldots + \frac{n}{2^{n-1}} + \ldots\right)$$

To compute this sum we start from the identity (2) and derivate both sides with respect to x. This gives us the following identity:

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \frac{1}{(1 - x)^2}$$

This is an important sum: remember how we obtained it even if you forget the end result. Now we can figure out how many pizzas Bob eats:

$$\frac{1}{2}\left(1+\frac{2}{2}+\frac{3}{2^2}+\ldots+\frac{n}{2^{n-1}}+\ldots\right) \;=\; \frac{1}{2}\frac{1}{(1-\frac{1}{2})^2}=2$$

Thus, given an infinite amount of time, Alice eats one pizza and Bob eats two pizzas.

Rule of Thumb You should be able to compute any variation on the geometric series Eq.(1) by applying the derivative.

Recurrences

Sums in Disguise

Don't be fooled by these:

$$\begin{aligned} f_0 &= 0 \\ f_n &= f_{n-1}+3n \end{aligned}$$

This is not really a recurrence, but a sum:

$$f_n \;=\; 3+3\cdot 2+3\cdot 2+\ldots+3\cdot n=3\frac{n(n+1)}{2}$$

Fibonacci

The Fibonacci sequence is a real recurrence:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1}+f_{n-2} \end{aligned}$$

We want to compute a closed formula for f_n. Here we'll just make a wild guess. We will guess that $f_n = a^n$, for some constant a. What is the constant a ? Substitute in $f_n = f_{n-1} + f_{n-2}$ and we obtain:

$$\begin{aligned}
a^n &= a^{n-1} + a^{n-2} \\
a^2 &= a + 1 \\
a^2 - a - 1 &= 0
\end{aligned}$$

The last equation is called the *characteristic equation* of the recurrence. Its roots are $a_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. Thus, have two choices for a; plus, we can always multiply a^n with some constant. It follows that the general expression for f_n is:

$$f_n = c_1 \cdot a_1^n + c_2 \cdot a_2^n \tag{3}$$

Exercise: prove that, if a_1, a_2 are the roots of the equation $x^2 - x - 1 = 0$ and c_1, c_2 are any two constants, then f_n given by the expression (3) satisfies the recurrence: $f_n = f_{n-1} + f_{n-2}$ forall $n \geq 2$.

We already know a_1, a_2, but how do we compute c_1, c_2 ? Simply by knowing f_0 and f_1:

$$\begin{aligned}
f_0 = 1 &= c_1 a_1^0 + c_2 a_2^0 = c_1 + c_2 \\
f_1 = 1 &= c_1 a_1^1 + c_2 a_2^1 = c_1 a_1 + c_2 a_2
\end{aligned}$$

We solve for c_1, c_2:

$$\begin{aligned}
c_1 &= \frac{a_2 - 1}{a_2 - a_1} = \frac{\sqrt{5} - 1}{2\sqrt{5}} \\
c_2 &= \frac{a_1 - 1}{a_2 - a_1} = \frac{-\sqrt{5} - 1}{2\sqrt{5}}
\end{aligned}$$

We can write these two constants more elegantly, by noticing:

$$\begin{aligned}
\frac{1}{a_1} &= \frac{1}{1 + \sqrt{5}} = -\frac{1 - \sqrt{5}}{2} \\
\frac{1}{a_2} &= \frac{1}{1 - \sqrt{5}} = -\frac{1 + \sqrt{5}}{2}
\end{aligned}$$

thus;

$$
\begin{aligned}
c_1 &= -\frac{1}{a_1\sqrt{5}} \\
c_2 &= \frac{1}{a_2\sqrt{5}}
\end{aligned}
$$

Putting everything together we obtain:

$$
f_n = \frac{1}{\sqrt{5}}\left(a_2^{n-1} - a_1^{n-1}\right)
$$

Other Recurrences

Other recurrences "like the Fibonacci series" are solved similarly.

$$
\begin{aligned}
f_0 &= 6 \\
f_1 &= 10 \\
f_2 &= 20 \\
f_n &= 6f_{n-1} - 11f_{n-2} + 6f_{n-3}
\end{aligned}
$$

We proceed in the same way. We make a wild guess that $f_n = a^n$, and search for the constant a that satifies the recurrence $f_n = 6f_{n-1} - 11f_{n-2} + 6f_{n-3}$:

$$
\begin{aligned}
a^n &= 6a^{n-1} - 11a^{n-2} + 6a^{n-3} \\
a^3 &= 6a^2 - 11a + 6 \\
a^3 - 6a^2 + 11a - 6 &= 0
\end{aligned}
$$

Here we have three roots: $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$, and therefore the general expression for f_n is:

$$
f_n = c_1 1^n + c_2 2^n + c_3 3^n
$$

To find the three constants c_1, c_2, c_3 we build a system of equations by using the initial conditions f_0, f_1, f_2:

$$\begin{aligned}
f_0 = 6 &= c_1 + c_2 + c_3 \\
f_2 = 10 &= c_1 + 2c_2 + 3c_3 \\
f_3 = 20 &= c_1 + 4c_2 + 9c_3
\end{aligned}$$

The solutions are: $c_1 = 3$, $c_2 = 2$, $c_3 = 1$, and therefore, f_n is given by:

$$f_n = 3 + 2^{n+1} + 3^n$$

Roots with Multiplicity > 1

If the characteristic equation has a root with multiplicity > 1, then we need to do this. Consider:

$$\begin{aligned}
f_0 &= 2 \\
f_1 &= 7 \\
f_n &= 6f_{n-1} - 9f_{n-2}
\end{aligned}$$

Trying $f_n = a^n$, we are lead to the characteristic equation $a^2 - 6a + 9 = 0$, which has a single root, with multiplicity 2: $a_1 = a_2 = 3$. We cannot apply blindly the method above, because we don't have two constants c_1, c_2 to satisfy the two initial conditions f_0, f_1: the system $c_1 + c_2 = f_0$, $c_1 a_1 + c_2 a_2 = f_1$ is underdefined when $a_1 = a_2$. However, when the characteristic equation has a root a_1 with multiplicity 2, then $f_n = c_1 a_1^n + c_2 n a_1^{n-1}$ also satisfies the recurrence (we have multiplied with n the second term). Please prove by induction on n that this expression for f_n satisfies the recurrence. Now we can solve for c_1, c_2 and obtain:

$$f_n = 2 \cdot 3^n + n \cdot 3^{n-1}$$

In general, if a_i is a root with multiplicity k, then we include in f_n all the terms a_i^n, na_i^{n-1}, $n(n-1)a_i^{n-2}$, $\ldots$, $n(n-1)\cdots(n-k+1)a_i^{n-k+1}$.