Context-free Grammars and Languages

Context-free languages

$\Sigma = \{a, +, *, (,)\}$

Rules

$E \to P + E$

$E \to P * E$

$E \to P$

$P \to (E)$

$P \to a$

Example strings in L(G)

a

(a)

A *derivation tree* or *parse tree* in G

$\to a + (a)$

Another string in L(G)

A CFG $G = (V, \Sigma, R, S)$

V a finite set ("variables")

$V \cap \Sigma = \emptyset$

("start" or "sentence") $S \in V$

("rules") R is a finite subset of $V \times (V \cup \Sigma)^*$

Notation

$\rightarrow$ in rules (only; "produces" or "may be rewritten as")

$\Rightarrow$ "yeilds" :

relation on strings in $(V \cup \Sigma)^*$

$\alpha A \beta \Rightarrow \alpha \gamma \beta$

if $A \rightarrow \gamma$ is a rule

for all $\alpha, \beta \in (V \cup \Sigma)^*$

i.e., "context-free"

$\Rightarrow^*$ "derives" (reflexive, transitive closure of $\Rightarrow$; "0 or more steps")

$\alpha \Rightarrow^* \beta$ means $\exists$ $\alpha_0, \alpha_1, \cdots \alpha_k$ $k \geq 0$

$\alpha = \alpha_0 \Rightarrow \alpha_1 \Rightarrow \alpha_2 \cdots \alpha_k = \beta$

$L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^* w \}$

Example

$G = (V, \Sigma, R, S)$

$V = \{S\}$

$\Sigma = \{a, b\}$

R:

$S \to aSb \mid \varepsilon$

$S \Rightarrow \varepsilon$

$S \Rightarrow aSb \Rightarrow ab$

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$

$\vdots$

$L(G) = \{ a^n b^n \mid n \geq 0 \}$

Note that L(G) is non-regular

$G_2 = (V, \Sigma, R, S)$

$V = \{S\}$

$\Sigma = \{a, b\}$

R:

$S \to aSa \mid bSb \mid \varepsilon$

$S \Rightarrow \varepsilon$

$S \Rightarrow aSa \Rightarrow aa$

$S \Rightarrow bSb \Rightarrow bb$

$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$

$L(G_2)$ = even length palindromes

$\{ww^R \mid w \in \Sigma^*\}$

Example

We'll see later that

$$L_{two} = \{ww \mid w \in \Sigma^*\}$$

is *not* context free. At first glance, you might think that adding a new start symbol S' and a rule

$$S' \to SS$$

to G_2 would generate L_{two}, but it doesn't; it generates all strings in L_{two} *plus* many others, since derivations from the two S's are *not coordinated*. (Why not? It's *context-free*; what happens to one S can't influence the other.)

Example

G_3: as above but add

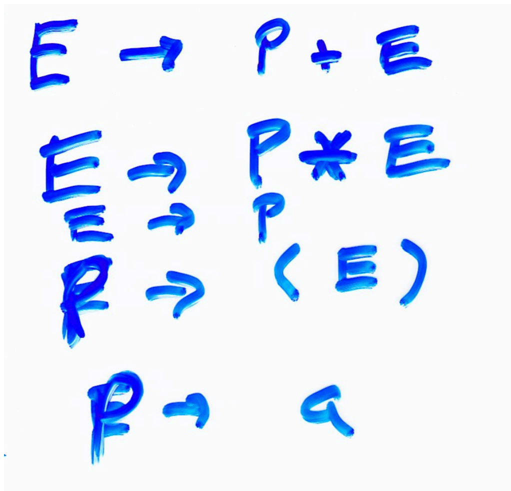
$S \to a \mid b$

$L(G_3)$ <u>all</u> palindromes

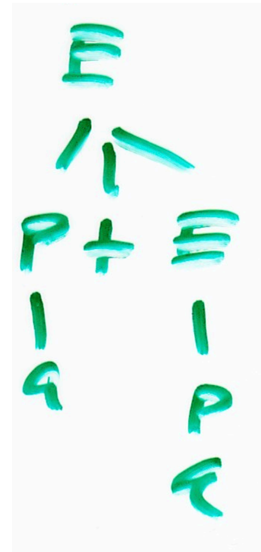
$\{ w \in \Sigma^* \mid w = w^R \}$

Trees, Derivations and Ambiguity

A grammar



A tree



3 derivations correspond to same tree (same rules being used in the same places, just written in different orders in the linear derivation)

1) E => P+E => a+E => a+P => a+a

2) E => P+E => P+P => a+P => a+a

3) E => P+E => P+P => P+a => a+a

But only one *leftmost* derivation corresponds to it

(and *vice versa*). (more in HW?)

Another grammar for the same language:

$$E \rightarrow E{+}E \mid E{*}E \mid (E) \mid a$$

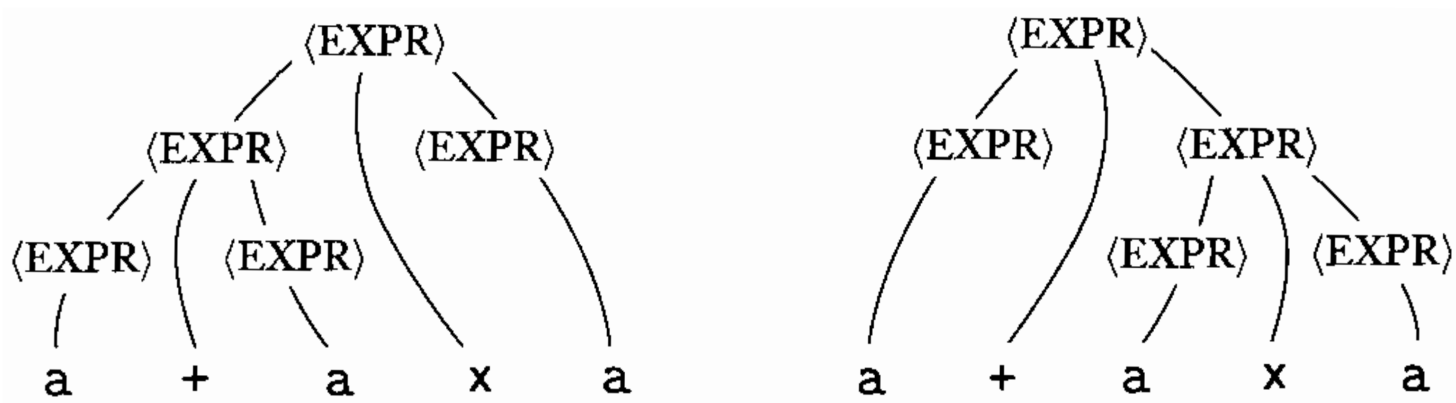


Fig 2.6: Two parse trees for a+a×a in grammar G_5

This grammar is *ambiguous:* there is a string in $L(G_5)$ with two *different* parse trees, or, equivalently, with 2 different leftmost derivations. Note the pragmatic difference: in general, (a+a)*a != a+(a*a); which is "right"?

$E \to E+E \mid E*E \mid a$

ambiguous

$\neq$

$E \Rightarrow E+E$
$\Rightarrow a+E$
$\Rightarrow a+E*E$
$\Rightarrow a+a*E$
$\Rightarrow a+a*a$

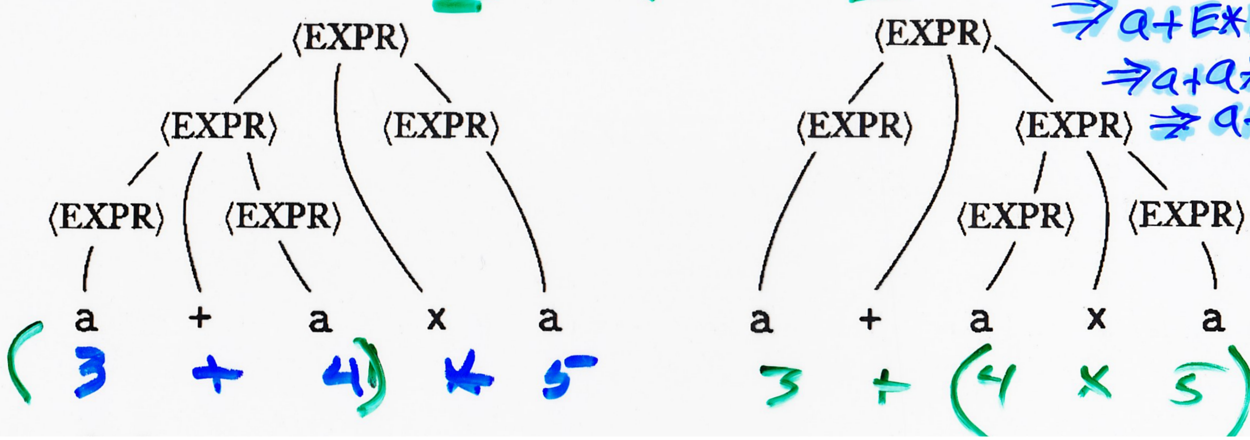


Fig 2.6: Two parse trees for a+a×a in grammar G_5

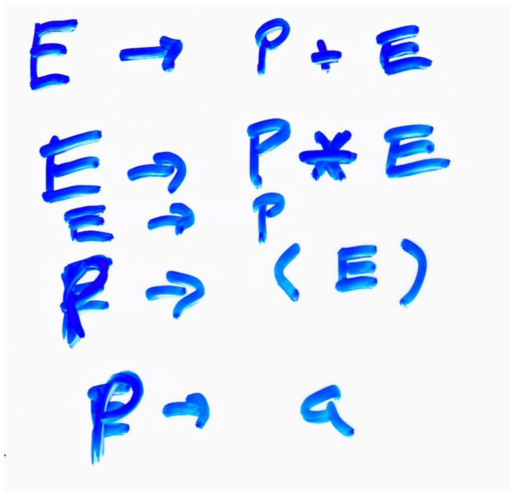
Leftmost deriv

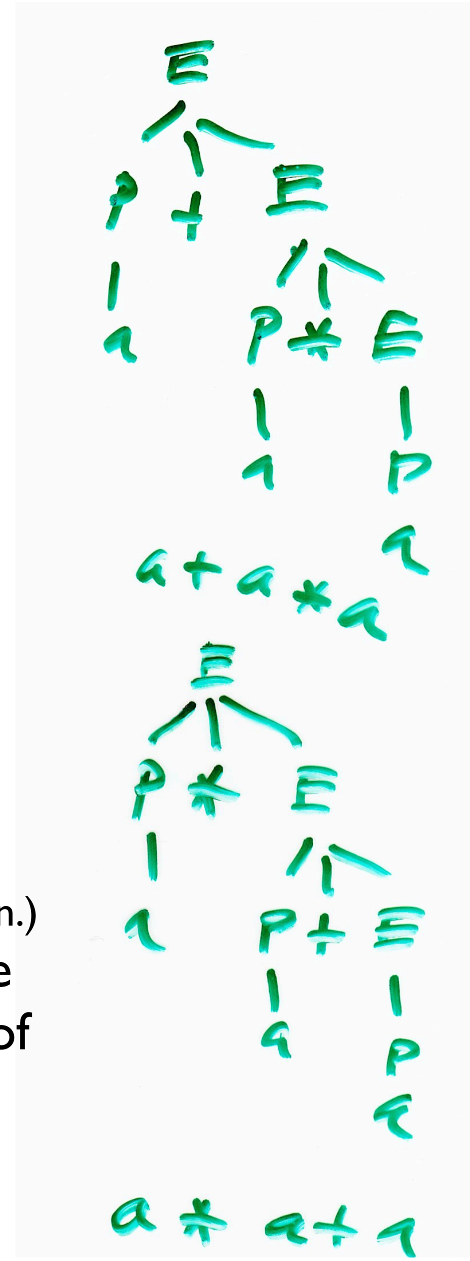
$E \Rightarrow_L E*E \Rightarrow_L E+E*E \Rightarrow_L a+E*E$
$\Rightarrow_L a+a*E$
$\Rightarrow_L a+a*a$

non-leftmost deriv

$E*E \Rightarrow E*a \Rightarrow E+E*a \Rightarrow E+a*a$

The “E, P” grammar again





This grammar is *un*ambiguous.

(Why? Very informally, the 3 E rules generate $P((\text{‘+’}\cup\text{’*’})P)^*$ and only via a parse tree that “hangs to the right”, as shown.)

But it has another undesirable feature: Parse tree structure does not reflect the usual precedence of * over +. E.g., tree at lower right suggests “a * a + a == a * (a + a)”

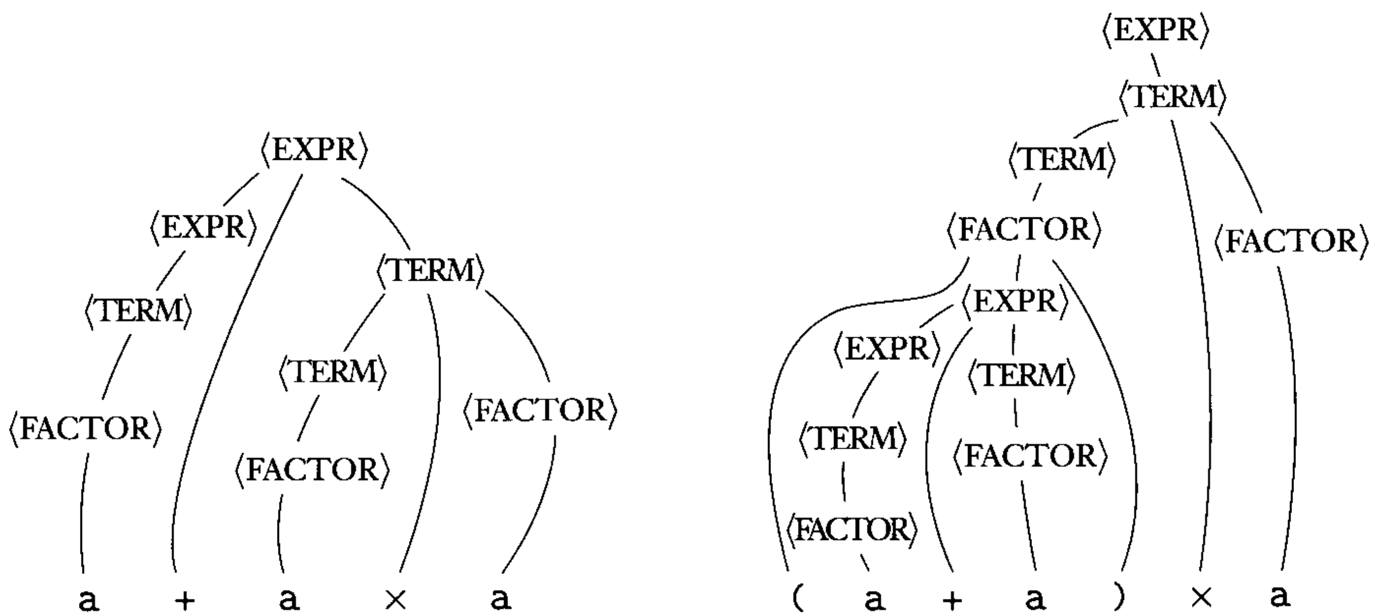
EXAMPLE 2.4

Consider grammar $G_4 = (V, \Sigma, R, \langle \text{EXPR} \rangle)$.

V is $\{\langle \text{EXPR} \rangle, \langle \text{TERM} \rangle, \langle \text{FACTOR} \rangle\}$ and Σ is $\{\mathtt{a}, +, \times, (,)\}$. The rules are

$$\begin{aligned}
\langle \text{EXPR} \rangle &\to \langle \text{EXPR} \rangle + \langle \text{TERM} \rangle \mid \langle \text{TERM} \rangle \\
\langle \text{TERM} \rangle &\to \langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle \mid \langle \text{FACTOR} \rangle \\
\langle \text{FACTOR} \rangle &\to (\langle \text{EXPR} \rangle) \mid \mathtt{a}
\end{aligned}$$

The two strings a+a×a and (a+a)×a can be generated with grammar G_4. The parse trees are shown in the following figure.



A more complex grammar, again the same language. This one is unambiguous *and* its parse trees reflect usual precedence/associativity of plus and times.

$$L = \{ \underbrace{a^i b^j}_{A} c^k \mid i=j \text{ or } j=k \}$$

$$\begin{array}{l} S \to AC \mid DB \\ A \to aAb \mid \varepsilon \\ C \to cC \mid \varepsilon \\ D \to aD \mid \varepsilon \\ B \to bBc \mid \varepsilon \end{array} \Bigg\} \; G$$

$a^{10} b^{10} c^{22}$

$a^{10} b^{22} c^{22}$

$a^{10} b^{10} c^{10}$

Can we always tweak the grammar to make it unambiguous?

No! Language L is a CFL; grammar at left. Easy to see this G is ambiguous–strings of the form $a^n b^n c^n$ can come from the i=j (AC) or j=k (DB) path. Hard to prove, but true, that *every G for this L is also ambiguous*. Intuitively, a grammar can only match a's & b's or b's & c's, not both. As a related point, { $a^n b^n c^n$ | n>0 } is *not* CFL.

G is ambiguous

L is *inherently ambiguous*, meaning every G for L is ambiguous

Some closure results for CFLs

Theorem:

The set of context-free languages is closed under $\cup$, $\bullet$, and $*$

Corollary:

All regular languages are CFLs

Proof Sketch:

Directly give simple CFLs for $\varnothing$, $\{\varepsilon\}$, and $\{a\}$ for each $a \in \Sigma$. Combine them using the above theorem.

(Aside:

We'll later prove that CFLs are *not* closed under intersection or complementation.)

Proof: Closure under Concatenation

$G_i = (V_i, \Sigma, R_i, S_i)$

be 2 CFG's

with $V_1 \cap V_2 = \emptyset$

let $S \notin V_1 \cup V_2$

Build new grammar

$$G = (V, \Sigma, R, S)$$

$$V = V_1 \cup V_2 \cup \{S\}$$

$$R = R_1 \cup R_2 \cup \{S \to S_1 S_2\}$$

$\forall x \in L_1 \ \forall y \in L_2$

$$S_1 \overset{*}{\underset{G_1}{\Rightarrow}} x \ \ \& \ \ S_2 \overset{*}{\underset{G_2}{\Rightarrow}} y$$

$$\therefore \ S \underset{G}{\Rightarrow} S_1 S_2 \overset{*}{\underset{G}{\Rightarrow}} x S_2 \overset{*}{\underset{G}{\Rightarrow}} x y$$

$$\therefore \ L_1 \cdot L_2 \subseteq L(G)$$

Suppose $S \overset{*}{\Rightarrow}_G w$

$* \quad \boxed{S \Rightarrow_G S_1 S_2 \overset{*}{\Rightarrow}_G w}$

Then, for some $x, y \in \Sigma^*$

$S_1 S_2 \overset{*}{\Rightarrow}_L x S_2$

using only rules from G_1

$x S_2 \overset{*}{\Rightarrow}_L xy = w$

using only G_2 rules

$$\left.\begin{array}{ll} S_1 \overset{*}{\Rightarrow} x & \text{in } G_1 \\ S_2 \Rightarrow y & \text{in } G_2 \end{array}\right] **$$

$L(G) \subseteq L_1 \cdot L_2$

A key issue in this direction of the proof is that, since $V_1 \cap V_2 = \varnothing$, there is no "crosstalk" between the two sub-grammars: any derivation in G from S_1 is also a derivation in G_1, and likewise S_2/G_2, so derivation (*) above in G can be split into (**) in G_1 & G_2.