Theorem

CFL's are closed under

$\cup$, $\circ$, $*$

Corr.

all regular languages are CFL's.

Pf: Give CFL's for $\emptyset$, $\{\varepsilon\}$, $\{a\}$ for each $a \in \Sigma$

Concat

$G_i = (V_i, \Sigma, R_i, S_i)$

be 2 CFG's

with $V_1 \cap V_2 = \phi$

let $S \notin V_1 \cup V_2$

Build new grammar

$$G = (V, \Sigma, R, S)$$

$$V = V_1 \cup V_2 \cup \{S\}$$

$$R = R_1 \cup R_2 \cup \{S \to S_1 S_2\}$$

$\forall x \in L_1 \; \forall y \in L_2$

$$S_1 \Rightarrow^* x \quad \& \quad S_2 \Rightarrow^* y$$

$$\therefore \; S \underset{L}{\Rightarrow} S_1 S_2 \underset{L}{\Rightarrow^*} x S_2 \underset{L}{\Rightarrow^*} xy$$

$\therefore \; L_1 \cdot L_2 \subseteq L(G)$

Suppose $S \overset{*}{\Rightarrow} w$

$*$ $\boxed{S \Rightarrow S_1 S_2 \overset{*}{\Rightarrow} w}$ for some $x \in \Sigma^*$

$S_1 S_2 \overset{*}{\Rightarrow}_L x S_2$

using only rules from G_1

$x S_2 \overset{*}{\Rightarrow}_L x y = w$

using only G_2 rules

$S_1 \overset{*}{\Rightarrow} x$ in G_1

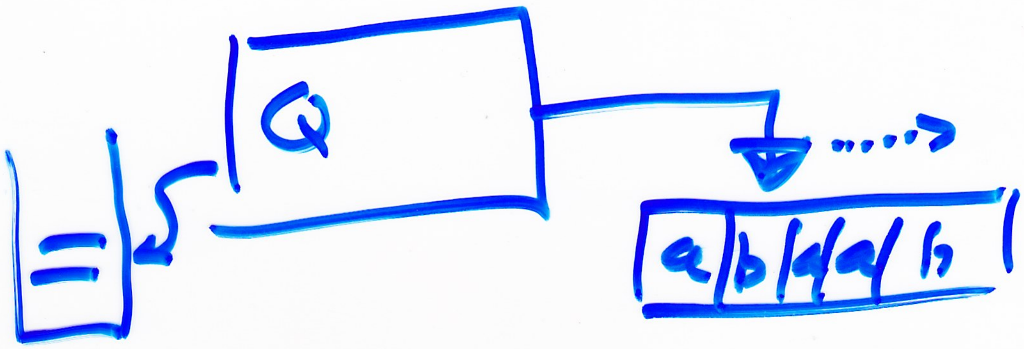
$S_2 \Rightarrow y$ in G_2 $\Big]$ $**$

$L(G) \subseteq L_1 \cdot L_2$

A key issue in this direction of the proof is that, since $V_1 \cap V_2 = \emptyset$, any derivation in G from S_1 is also a derivation in G_1, and likewise S_2/G_2, so derivation in G $*$ above can be split into $**$ in G_1 & G_2.

CF but not Regular

$\underline{a^n b^n}$, ww^R, $\#a = \#b$, ...



Recursive

$S \to aSb \mid \varepsilon$

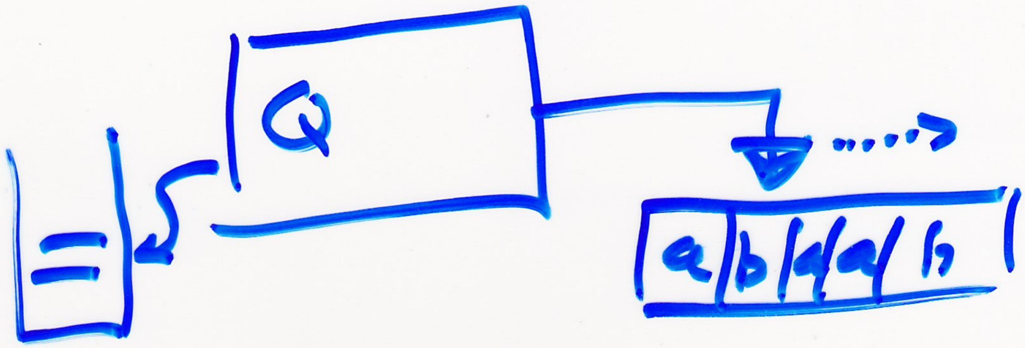
Intuit

$a^n b^n$: push a's
pop/match b's

2.2.04

CF but not Regular

$a^n b^n$, ww^R, $\#a = \#b$, ...



Recursive

$S \to aSb \mid \varepsilon$

$S \to aSa \mid bSb \mid \varepsilon$

Intuit

$a^n b^n$: push a's
pop/match b's

ww^R : push input
At middle, (Guess!)
Flip state to pop/match

22.4,5

Pushdown Automaton

$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$

Q is finite set (states)

Σ . . . . (input alphabet)

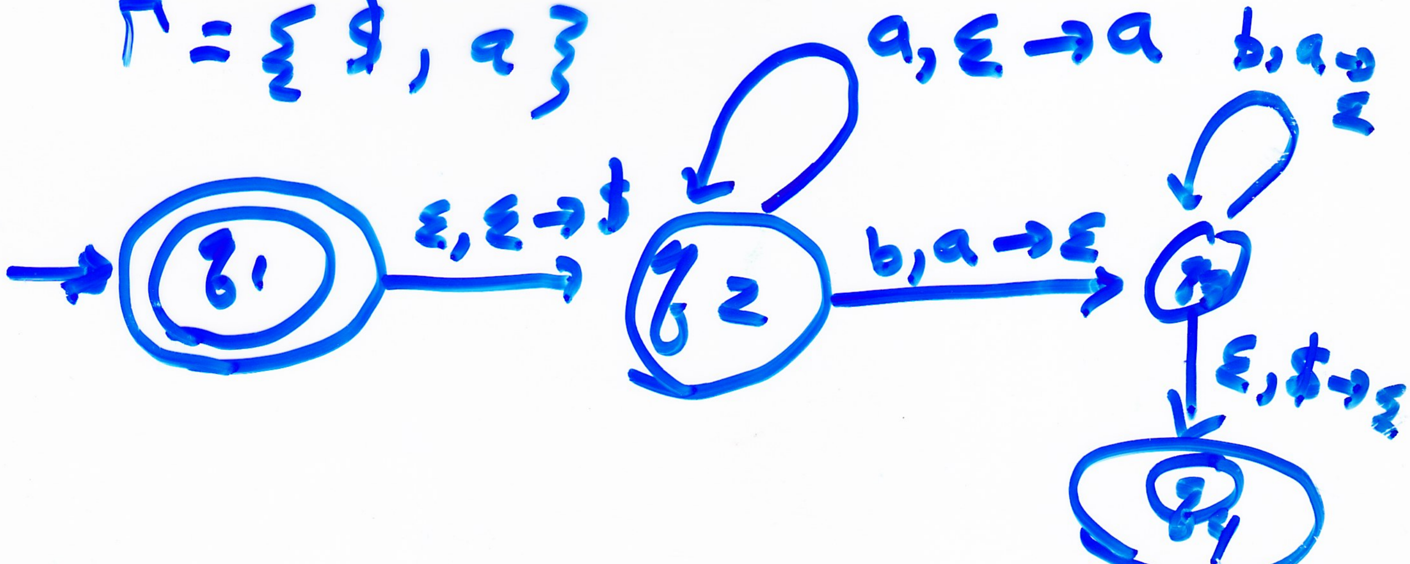
Γ . . . . (stack alphabet)

$q_0 \in Q$ start state

$F \subseteq Q$ accept states

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to 2^{Q \times \Gamma_\varepsilon}$$

$\Gamma = \{ \$, a \}$



22.6

M can reach state q with $\gamma \in \Gamma^*$ on its stack after reading w

if $\exists\, w_1 w_2 \dots w_m \in \Sigma_\varepsilon$

st $w = w_1 \cdot w_2 \dots w_m$

$\exists\, r_0\, r_1 \dots r_m \in Q$

$\exists\, s_0, \dots s_m \in \Gamma^*$

(1) $r_0 = q_0$,

(2) $s_0 = \varepsilon$

(3) $\forall\, i = 0 .. m-1$

$(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$

for some $a, b \in \Gamma_\varepsilon$, $t \in \Gamma^*$

with $s_i = at$, $s_{i+1} = bt$

(4) $\gamma = s_m$

M accepts w if $r_m \in F$

$L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$

Example: A computation of M above on input $w = aabb$

State	Stack	remaining input	
$r_0 = q_1$	$s_0 = \varepsilon$	$aabb$	
			$(q_2, \$) \in \delta(q_1, \varepsilon, \varepsilon)$
$r_1 = q_2$	$s_1 = \$$	$aabb$	
			$(q_2, a) \in \delta(q_2, a, \varepsilon)$
$r_2 = q_2$	$s_2 = \$a$	abb	
			" "
$r_3 = q_2$	$s_3 = \$aa$	bb	
			$(q_3, \varepsilon) \in \delta(q_2, b, a)$
$r_4 = q_3$	$s_4 = \$a$	b	
			$(q_3, \varepsilon) \in \delta(q_3, b, a)$
$r_5 = q_3$	$s_5 = \$$	ε	
			$(q_4, \varepsilon) \in \delta(q_3, \varepsilon, \$)$
$r_6 = q_4$	$s_6 = \varepsilon$	ε	

top of stack @ right end

So, e.g., "M can reach q_3 w/ \$ on stack reading a^2b^2" and "M can reach q_4 w/ ε stack reading a^2b^2" and "M accepts a^2b^2".