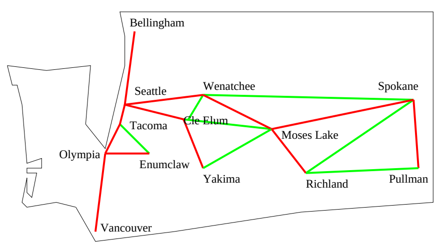
17—Minimum Spanning Trees and Kruskal's Algorithm

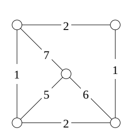
May 19, 2002

Subgraphs

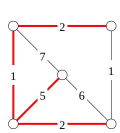


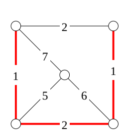
- Cities and edges form a *graph*
- Cities and red edges form a *subgraph*
- *Weighted* graphs have a value: value$(G) = \sum_{e \in E}$ weight(e)

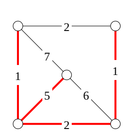
A Graph Problem



G: a weighted graph



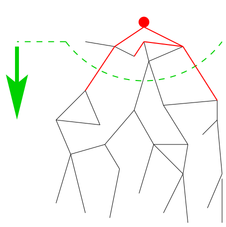


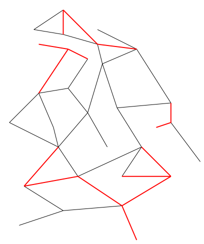


What is the *cheapest, connected* subgraph of G?

How to Find Spanning Trees

Two *Greedy* Algorithms





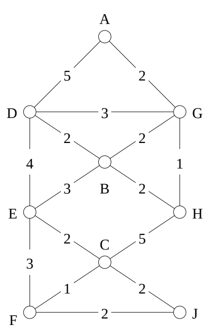
Prim's Algorithm

Almost the same as Dijkstra's

Kruskal's Algorithm

Totally different!

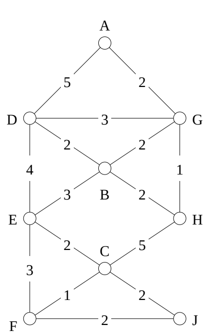
Kruskal's Algorithm (Almost)




```
Kruskal(Graph G)
{
   Graph MST;
   PQ pq.Build(G.edges());

   until (MST is a tree) {

     Edge e = pq.DeleteMin();


     MST.Add(e);

   }
   return MST;
}
```

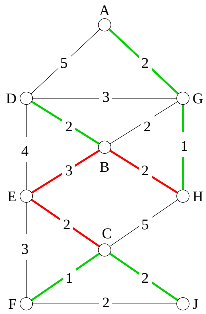
Key idea: We like short edges

Kruskal's Algorithm (Almost)



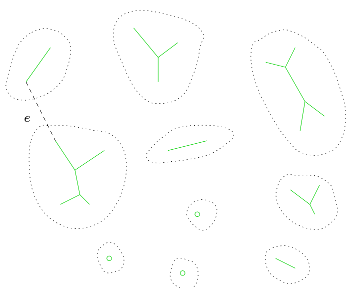

```
Kruskal(Graph G)
{
   Graph MST;
   PQ pq.Build(G.edges());

   while (num components > 1) {

     Edge e = pq.DeleteMin();

     if (e does not make a cycle)
       MST.Add(e);

   }
   return MST;
}
```

Why It Works



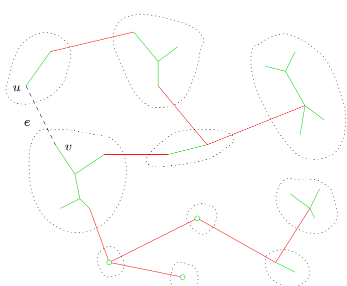
Can we always extend our current tree to some MST after choosing *any* valid minimum-cost edge?

A Proof



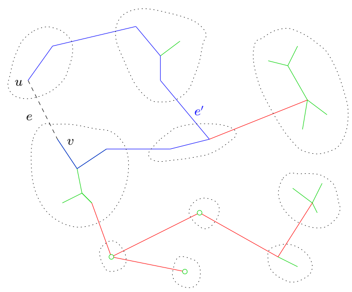
- G is our current forest
- e is the valid minimum-cost edge we're going to add
- Assuming there is an extension from G to a MST, how do we know e is in that extension?

Considering the Extension



- G is our current forest
- $e = (u, v)$ is the valid minimum-cost edge we're going to add
- F is a minimum-cost extension of G

More Slides = Better



- $\exists\ u \to v$ path p in F

  Why?

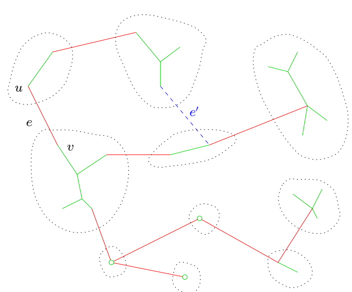
- p has an edge e' between components of G

  Why?

- weight(e) $\leq$ weight(e')

  Why?

Yes, More Slides = Better



- value($F - e' + e$) $\leq$ value(F)

  Why?

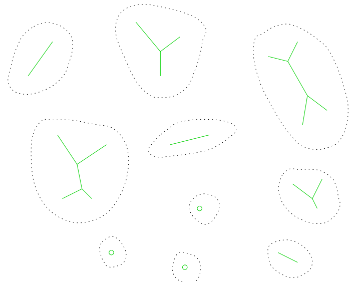
- $F - e' + e$ is a tree

  Why?

- Hence $G + e$ can be extended to an MST

  Yay!

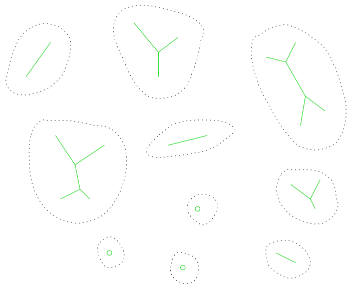
Implementing Kruskal's



```
Kruskal(Graph G)
{
   Graph MST;
   PQ pq.Build(G.edges());

   while (num components > 1) {

     Edge e = pq.DeleteMin();

     if (e does not make a cycle)
       MST.Add(e);

   }
   return MST;
}
```

Which part is hard?

Disjoint Sets



- Components are sets
- $e = (u, v)$ won't cause cycle if sets containing u and v are disjoint
- If we add e, union the sets containing u and v

Just like mazes

Kruskal Implementation

```
Kruskal(Graph G)
{
   Graph MST;
   PQ pq.Build(G.edges());
   DS ds.MakeSets(G.num_verticies());
   int components = G.num_verticies();

   while (                          ) {

     Edge e = pq.DeleteMin();

     DS::Set *u = ds.Find(e->u->Number());
     DS::Set *v = ds.Find(e->v->Number());

     if (                           ) {

       MST.Add(e);


     }

   }
   return MST;
}
```


Disjoint Set Implementation

```
class DS {

  class Set { ... };



public:
  class Set;
  void Union(Set *, Set *);
  Set *Find(Set *);

  void MakeSets(int n) {





  }

  Set *Find(int) {




  }
};
```

Running Time?

```
Kruskal(Graph G)
{
   Graph MST;
   PQ pq.Build(G.edges());
   DS ds.MakeSets(G.num_verticies());
   int components = G.num_verticies();

   while (components > 1) {

     Edge e = pq.DeleteMin();

     DS::Set *u = ds.Find(e->u->Number());
     DS::Set *v = ds.Find(e->v->Number());

     if (u != v) {
       MST.Add(e);
       ds.Union(u,v);
       components--;
     }

   }
   return MST;
}
```