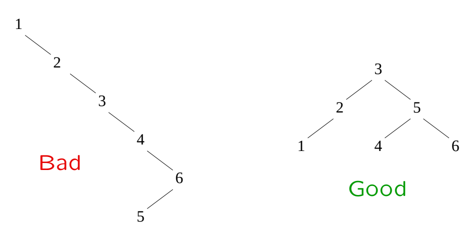
6: AVL Trees

CSE326 Spring 2002

April 11, 2002

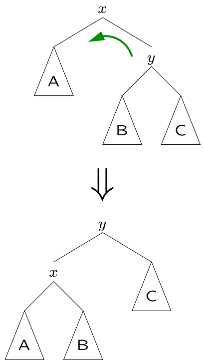
Balanced Trees



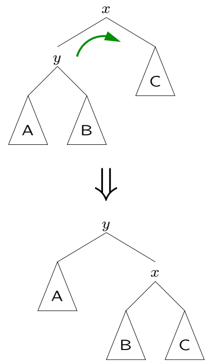
- *Same* data
- Different *tree structure*

Trees Containing 1, 2, 3, 4

General Restructuring Primitives

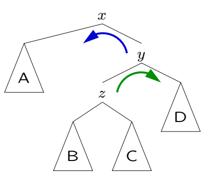


Rotate Left



Rotate Right

Double Rotation

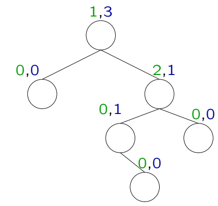


Strategy

Recipe for a Balanced Tree

1. Insert/Delete as for BST
2. Check if tree balanced
3. If not, use rotations to rebalance

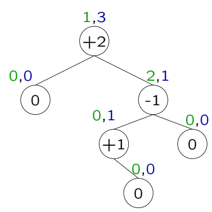
What is Balanced?



$\text{LeftHt}(v) = \begin{cases} 0 \\ 1 + \text{Height}(\text{LChild}(v)) \end{cases}$ $\text{RightHt}(v) = \begin{cases} 0 \\ 1 + \text{Height}(\text{RChild}(v)) \end{cases}$

A *balanced* tree has nearly the same height on both sides

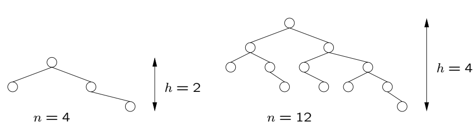
and AVL We Go!



- $\text{Balance}(v) = \text{RightHt}(v) - \text{LeftHt}(v)$
- T is an *AVL tree* iff every node $v \in T$ satisfies

$$-1 \leq \text{Balance}(v) \leq 1$$

Are AVL Trees Really Balanced?



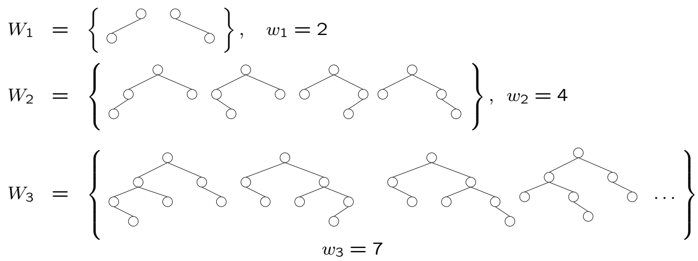
How bad can an AVL tree get?

- What's the worst h for a given n?
- What's the worst n for a given h?

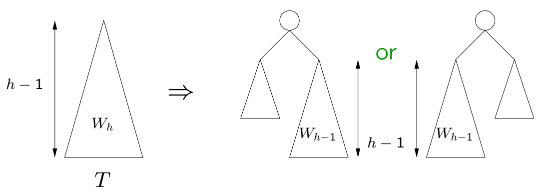
AVL Analysis

$$
\begin{aligned}
W_h &= \{\text{The smallest AVL trees of height } h\} \\
&= \{\text{All AVL trees with } w_h \text{ nodes of height } h\}
\end{aligned}
$$

$W_0 = \{\,\circ\,\}, \quad w_0 = 1$



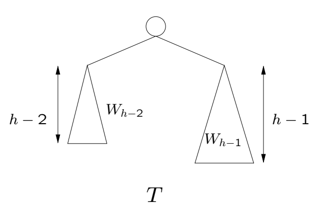
Worst of the Worst



Any $T \in W_h$ must have some W_{h-1} as a child tree

- If both height $< h-1$ or $> h-1$, then T not height h
- If child height $h-1$ but not in W_{h-1}, could get smaller tree by replacing it with $U \in W_{h-1}$

Still Bad

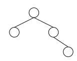


The other child *must* be $\in W_{h-2}$

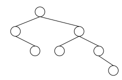
- Height either $h-1$ or $h-2$, or T not height h, or not AVL
- If height $h-1$ could make smaller by using $h-2$ instead
- If not $\in W_{h-2}$, could make smaller by replacing it with $U \in W_{h-2}$

How Bad is Bad?

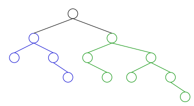
Trees in W_2, W_3 and W_4



$\in W_2$

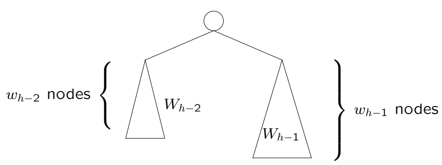


$\in W_3$



$\in W_4$

Bad to the Bone



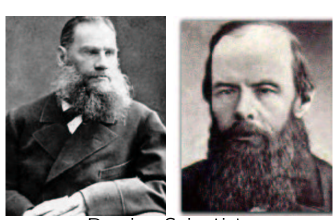
$T \in W_h$

$$
\begin{aligned}
w_h &= 1 + w_{h-2} + w_{h-1} \\
&= F_{h+3} - 1 = 2^{O(h)},
\end{aligned}
$$

where F_k is the k-th Fibbonacci number

Hence $h = O(\log n)$ for any AVL tree

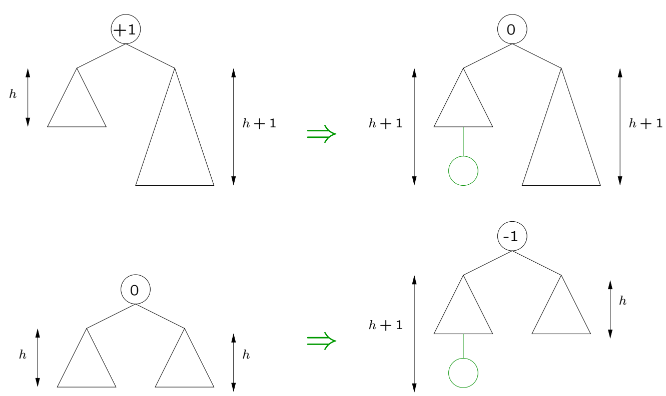
Background

- AVL $\equiv$ Adelson-Velskii and Landis
- 
  Russian Scientists
- Invent a data structure, your name will do down in history too!

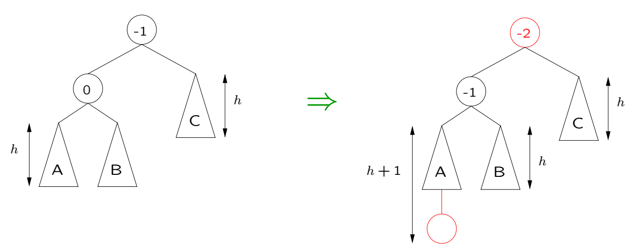
How Do We Insert?

- Insert on empty tree: $\wedge \Rightarrow$ 0
- We're off to a good start. . .

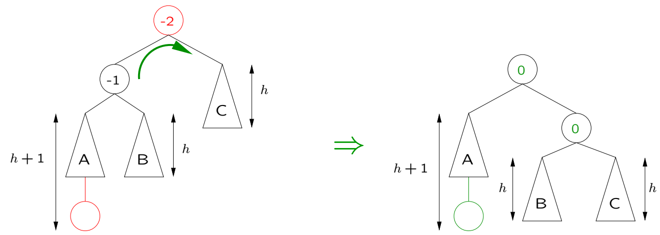
Easy Inductive Steps



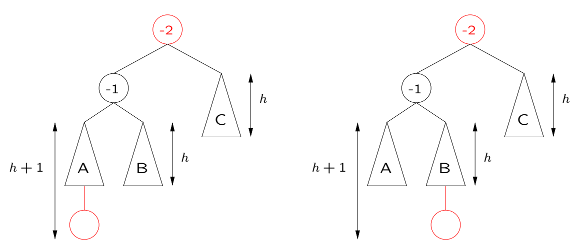
Oops



Rotate!



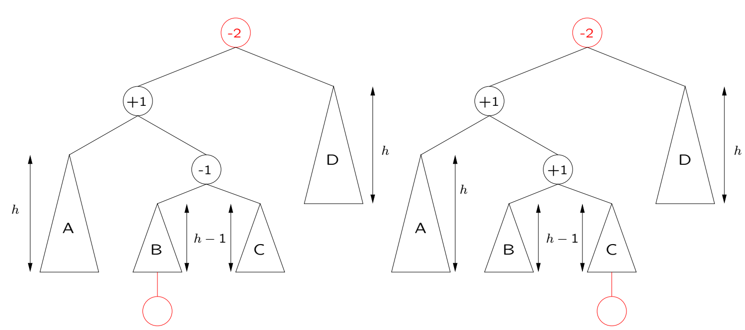
The Two Oops



Outside Case
(we just did this one)

Inside Case

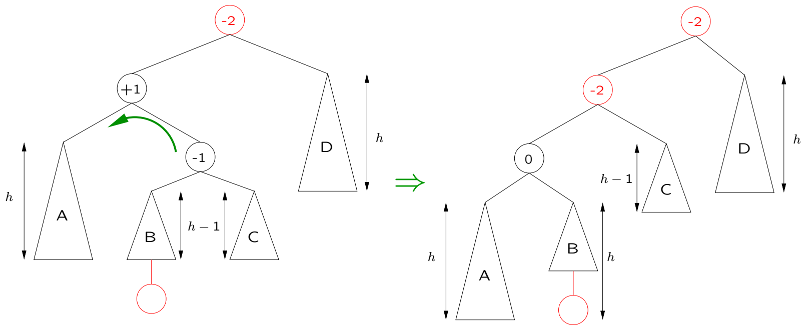
Inside Oops



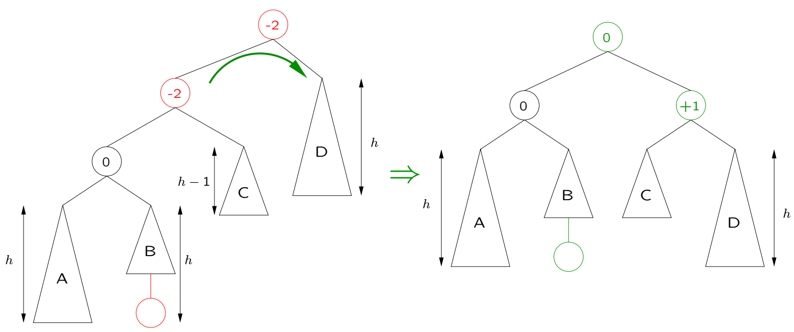
Case A

Case B

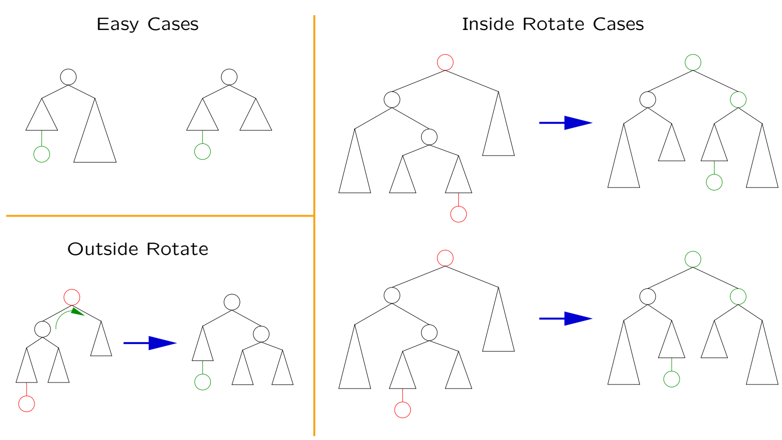
Rotate!



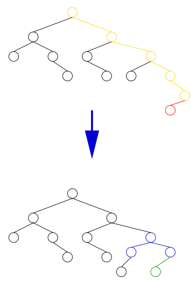
More! More!



What Happens to the Heights?

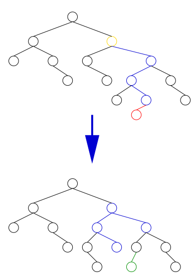


Performing an Insertion



1. Descend tree like BST insert
2. Insert Node
3. Retrace steps up tree, at each node
   (a) Update balance
   (b) Rotate if necessary

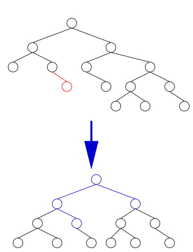
Analysis of Insertion



Consider *first* rotating node v

- Everything balanced *below* v
- Rotating at v makes Height(v) as it was *before* the insertion
- Hence nothing *above* v will see height change
- So we only need to rotate once!

Waving My Hands

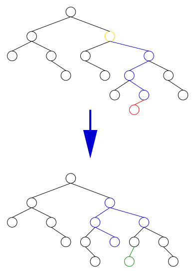


Deletion

1. Delete node as for BST, removing one node (either deleted, or inorder successor/predecessor)
2. If balance of removed's parent goes from $\pm 1 \rightarrow 0$, Height(p) changes and grandparent may become unbalanced
   - Rotate to rebalance
3. Grandparent's parent may now be unbalanced. . .
4. Continue back path to last balance $=$ 0 node

Running Time?

Summary



AVL Trees

- Find: $O(\log n)$
- Insert: $O(\log n)$
- Delete: $O(\log n)$