CSE 326: Data Structures

Topic #16: O Vertex, Where Art Thou?

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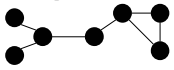
Today's Outline

Shortest path algorithms

1. Unweighted graphs: BFS
2. Weighted graphs without negative cost edges: Dijkstra's Algorithm
3. Negative cost edges but no negative cost cycles

Aside: Graph Connectivity

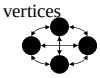
Undirected graphs are *connected* if there is a path between any two vertices



Directed graphs are *strongly connected* if there is a path from any one vertex to any other

Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*

A *complete* graph has an edge between every pair of vertices



The Shortest Path Problem

Given a graph G, edge costs $c_{i,j}$, and vertices s and t in G, find the shortest path from s to t.

For a path $p = v_0\ v_1\ v_2\ \ldots\ v_k$

– *unweighted length* of path $p = k$ (a.k.a. *length*)

– *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)

Path length equals path cost when ?

Single Source Shortest Paths (SSSP)

Given a graph G, edge costs $c_{i,j}$, and vertex s, find the shortest paths from s to <u>all</u> vertices in G.

– Is this harder or easier than the previous problem?

All Pairs Shortest Paths (APSP)

Given a graph G and edge costs $c_{i,j}$, find the shortest paths between <u>all pairs</u> of vertices in G.

– Is this harder or easier than SSSP?

– Could we use SSSP as a subroutine to solve this?

Variations of SSSP

- Weighted vs. unweighted
- Directed vs undirected
- Cyclic vs. acyclic
- Positive weights only vs. negative weights allowed
- Shortest path vs. longest path
- …

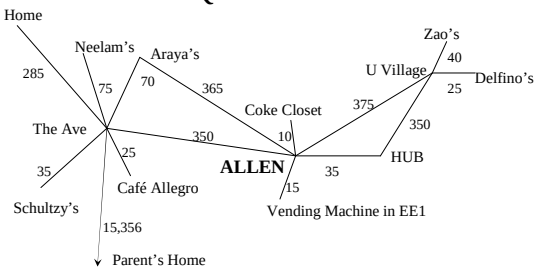
Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in construction management (see textbook)
- …

SSSP: Unweighted Version

Ideas?

Weighted SSSP: The Quest For Food



Can we calculate shortest distance to all nodes from Allen Center?

Dijkstra, Edsger Wybe

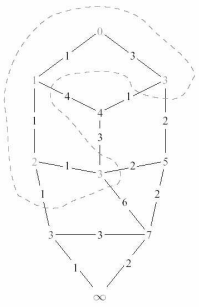
Legendary figure in computer science; was a professor at University of Texas.

Supported teaching introductory computer courses without computers (pencil and paper programming)

Supposedly wouldn't (until very late in life) read his e-mail; so, his staff had to print out messages and put them in his box.

E.W. Dijkstra (1930-2002)

Dijkstra's Algorithm: Idea

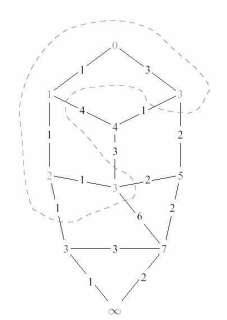


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Finished or known vertices
  - Shortest distance has been computed
- Unknown vertices
  - Have tentative distance

Dijkstra's Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

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Dijkstra's Algorithm: Pseudocode

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph
- Select an unknown node *b* with the lowest cost
- Mark *b* as known
- For each node *a* adjacent to *b*
  - *a*'s cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

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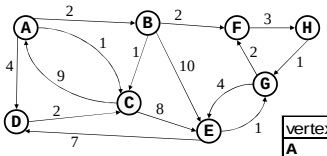
Dijkstra's Algorithm in Action



vertex	known	cost
A		
B		
C		
D		
E		
F		
G		
H		

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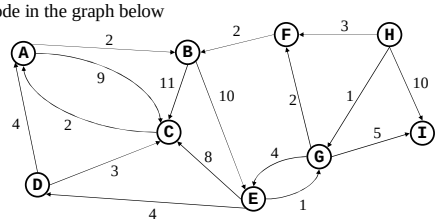
Time to play at home…

1. Use Dijkstra's algorithm to find the shortest path from **H** to every node in the graph below



2. Under what conditions will Dijkstra's algorithm fail?
3. What data structures should you use to best implement this algorithm? What running time does that yield?

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Dijkstra's Alg: Implementation

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph
- Select the unknown node *b* with the lowest cost
- Mark *b* as known
- For each node *a* adjacent to *b*
  - *a*'s cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

What data structures should we use?

Running time?

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Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs without negative weights
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Intuition for correctness:
  - shortest path from source vertex to itself is 0
  - cost of going to adjacent nodes is at most edge weights
  - cheapest of these must be shortest path to that node
  - update paths for new node and continue picking cheapest path

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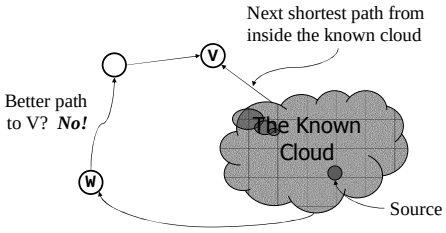
Correctness: The Cloud Proof



- If path to **V** is shortest, path to **W** must be at least as long
- So the path through **W** to **V** cannot be any shorter!

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node ***v*** (which is not in the cloud) is correct, we add it to the cloud

When does Dijkstra's algorithm not work?

Dijkstra's vs BFS

At each step:

1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

Dijkstra's Algorithm

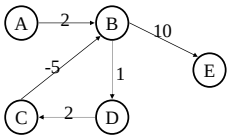
At each step:

1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

Breadth-first Search

Some Similarities:

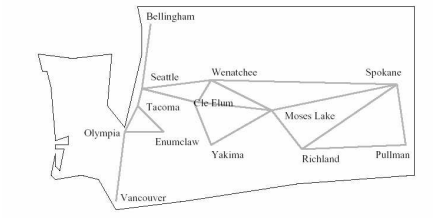
The Trouble with Negative Weight Cycles



What's the shortest path from A to E?

Problem?

Back to an Application: Moving Around Washington



What's the fastest way from Seattle to Spokane?

Answer:

To Do

- Read Chapter 9