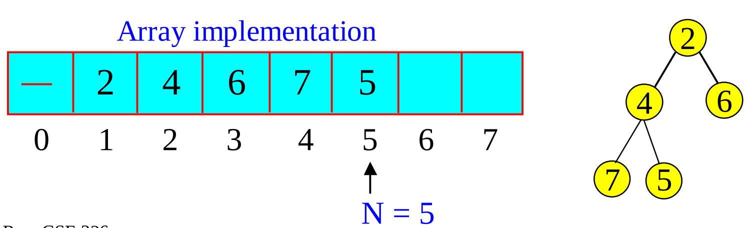
CSE 326 Lecture 11: Heaps and Binomial Queues

- What's on the menu today?
  - **Heaps**: DeleteMin, Insert, DecreaseKey, BuildHeap…
  - **Binomial Queues**: Merge, Insert, DeleteMin

- Covered in Chapter 6 in the text

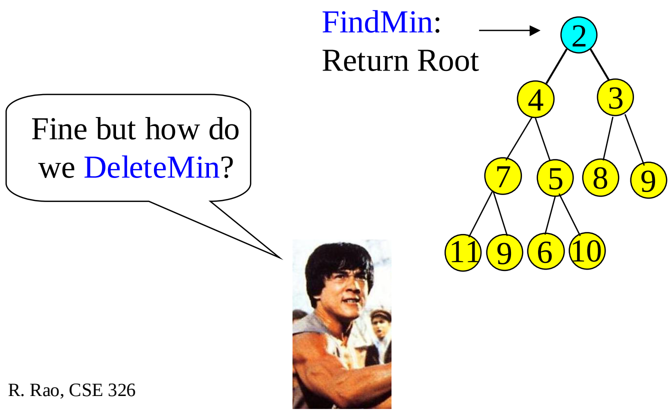
Flashback Definition of Heaps

- A binary heap is a binary tree that is:
  1. **Complete:** Tree completely filled except possibly the bottom level, which is filled from left to right
  2. **Satisfies the heap order property:** every node is smaller than (or equal to) its children
- Therefore, the root node is always the smallest in a heap

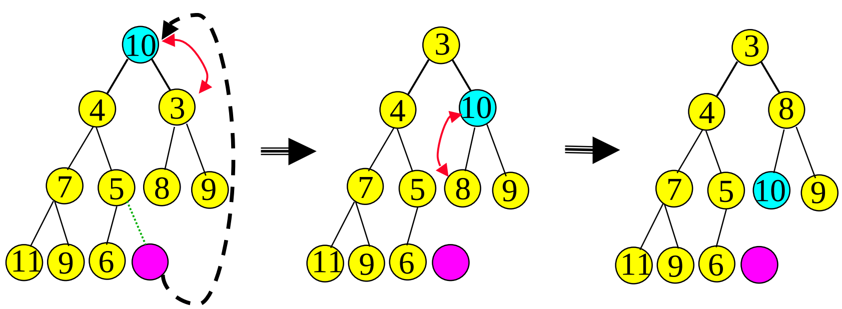


Last Time: Heap Operations

Basic Heap ADT Operations: FindMin, DeleteMin, Insert



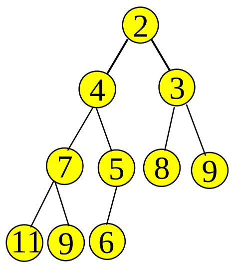
DeleteMin using Percolate Down



- Keep comparing with children A[2i] and A[2i + 1]
- Replace with smaller child and go down one level
- Done if both children are ≥ item or reached a leaf node
- Maintains both completeness and Heap order

Heaps: Insert Operation

Uh…How would I Insert 1 into this heap?

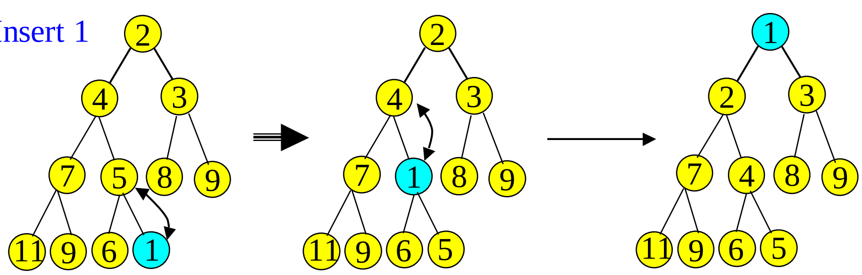


N = 10

—	2	4	3	7	5	8	9	11	9	6		

Insert: Percolate Up

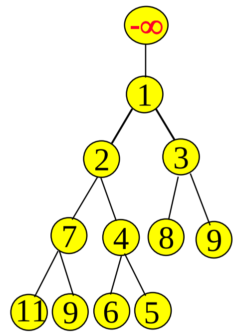
Insert 1



- Insert at <u>last node</u> and keep comparing with parent A[i/2]
- If parent larger, replace with parent and go up one level
- Done if parent ≤ item or reached top node A[1]
- Run time?

Sentinel Values

- Every iteration of Insert needs to test:
  1. if it has reached the top node A[1]
  2. if parent ≤ item
- Can avoid first test if A[0] contains a very large negative value (denoted by $-\infty$)
- Then, test #2 always stops at top
  - $-\infty$ < item for all items
- Such a data value that serves as a marker is called a sentinel
  - Used to improve efficiency and simplify code



A	-∞	1	2	3	7	4	8	9	11	9	6	5	

Summary of Heap ADT Analysis: Space

- Consider a heap of N nodes
- Space needed: O(N)
  - Actually, O(MaxSize) where MaxSize = size of the array
  - One more variable to store the current size N
  - With sentinel:
    Array-based implementation uses total N+2 space
  - Pointer-based implementation: pointers for children and parent
    - Total space = 3N + 1 (3 pointers per node + 1 for size)

Run Time Analysis of Heap ADT

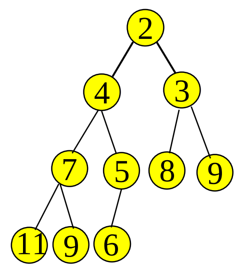
- Consider a heap of N nodes
- FindMin: O(1) time
- DeleteMin and Insert: O(log N) time
- BuildHeap from N inputs: What is the run time?
  - N Insert operations = O(N log N).
  - Can we do better?

Run Time Analysis of Heap ADT

- Consider a heap of N nodes
- FindMin: O(1) time
- DeleteMin and Insert: O(log N) time
- BuildHeap from N inputs: What is the run time?
  - N Insert operations = O(N log N).
  - Actually, can do better…O(N): Treat input array as a heap and fix it using percolate down
    - for i = N/2 to 1, percolateDown(i)
    - Why N/2? Nodes after N/2 are leaves!
    - See text for proof that this takes O(N) time.

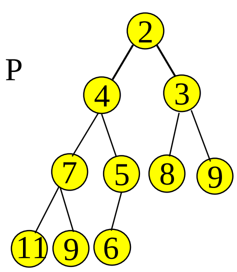
Other Heap Operations

- Find(X, H): Find the element X in heap H of N elements
  - What is the running time?
- FindMax(H): Find the maximum element in H
  - What is the running time?



One More Operation

- Find and FindMax: O(N)
- DecreaseKey(P,Δ,H): Decrease the key value of node at position P by a positive amount Δ.
  - E.g. System administrators can increase priority of important jobs.
  - How?
    - First, subtract Δ from current value at P
    - Heap order property may be violated
    - Percolate up or down?
    - Running time?



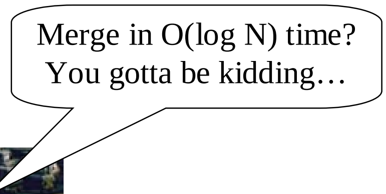
Some More Ops…

- DecreaseKey(P,Δ,H): Subtract Δ from current key value at P and percolate up. Running Time: O(log N)
- IncreaseKey(P,Δ,H): Add Δ to current key value at P and percolate down. Running Time: O(log N)
  - E.g. Schedulers in OS often decrease priority of CPU-hogging jobs (sound familiar?)
- Delete(P,H): E.g. Delete a job waiting in queue that has been preemptively terminated by user
  - How (using above operations)?
  - Running Time?

One Last Operation: Merge

- Delete(P,H): E.g. Delete a job waiting in queue that has been preemptively terminated by user
  - Use DecreaseKey(P,∞,H) followed by DeleteMin(H).
  - Running Time: O(log N)
- Merge(H1,H2): Merge two heaps H1 and H2 of size O(N). H1 and H2 are stored in two arrays. E.g. Combine queues from two different sources to run on one CPU.
  1. Can do O(N) Insert operations: O(N log N) time
  2. Better: Copy H2 at the end of H1 and use BuildHeap Running Time: O(N)

  Can we do even better? (i.e. Merge in O(log N) time?)

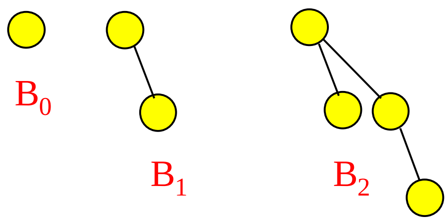


Say Hello to Binomial Queues

- Binomial queues support all three priority queue operations Merge, Insert and DeleteMin in O(log N) time
- Idea: Maintain a collection of heap-ordered trees
  - *Forest of binomial trees*
- Recursive Definition of Binomial Tree (based on height k):
  - Only one binomial tree for a given height
  - Binomial tree of height 0 = single root node
  - Binomial tree of height k = B_k = Attach B_{k-1} to root of another B_{k-1}

3 Steps to Building a Binomial Tree

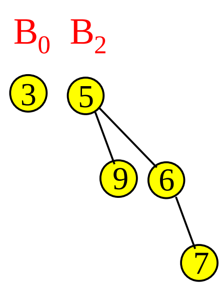
- To construct a binomial tree B_k of height k:
  1. Take the binomial tree B_{k-1} of height k-1
  2. Place another copy of B_{k-1} one level below the first
  3. Join the root nodes



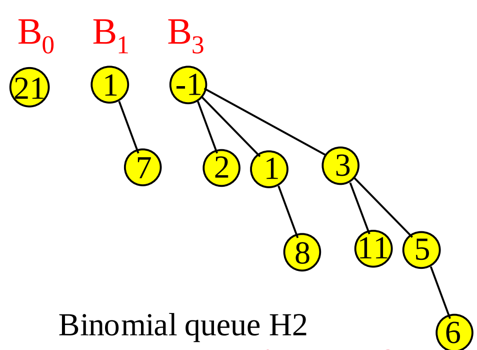
- Binomial tree of height k has exactly 2^k nodes (by induction)

Definition of Binomial Queues

Binomial Queue = "forest" of heap-ordered binomial trees



Binomial queue H1
5 elements = $2^0 + 2^2$
i.e. Uses B_0 and B_2



Binomial queue H2
11 elements = $2^0 + 2^1 + 2^3$
i.e. uses B_0 B_1 B_3

Binomial Queue Properties

- Suppose you are given a binomial queue of N nodes

1. There is a unique set of binomial trees for N nodes (express N in binary to find out which trees are in the set)
2. What is the maximum number of trees that can be in an N-node queue?
   - 1 node 1 tree B_0; 2 nodes 1 tree B_1; 3 nodes 2 trees B_0 and B_1; 7 nodes 3 trees B_0, B_1 and B_2 …

Number of Trees in a Binomial Queue

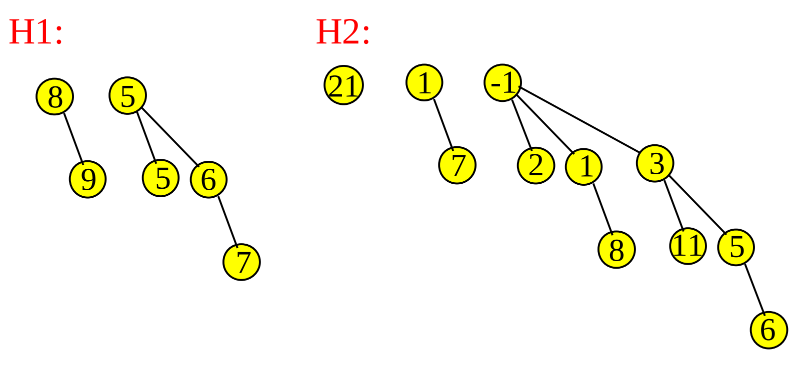
- What is the maximum number of trees that can be in an N-node binomial queue?
  - 1 node 1 tree B_0; 2 nodes 1 tree B_1; 3 nodes 2 trees B_0 and B_1; 7 nodes 3 trees B_0, B_1 and B_2 …
- Trees $B_0, B_1, \ldots, B_k$ can store up to $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ nodes = N.
- Maximum is when all k+1 trees are used.
- So, number of trees in an N-node binomial queue is $\leq$ k+1 = (log(N+1)-1)+1= O(log N)

Binomial Queues: Merge

- Main Idea: Merge two binomial queues by merging individual binomial trees
  - Since B_{k+1} is just two B_k's attached together, merging trees is easy
- Creating new queue by merging:
  1. Start with B_k for smallest k in either queue.
  2. If only one B_k, add B_k to new queue and go to next k.
  3. Merge two B_k's to get new B_{k+1} by making larger root the child of smaller root. Go to step 2 with k = k + 1.

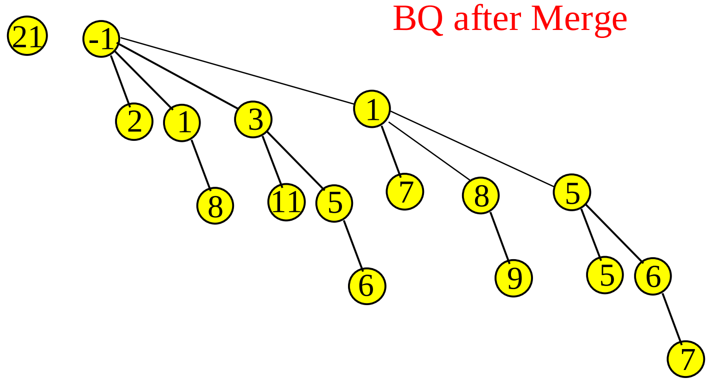
Binomial Queues: Merge Exercise

- What do you get when you Merge H1 and H2?



Binomial Queues: Merge

BQ after Merge

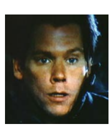


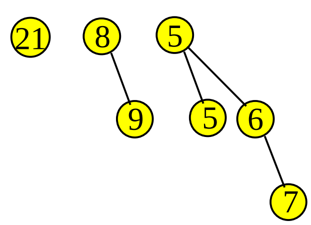
- What is the run time for Merge of two O(N) queues?

Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
  - Keep connecting roots of trees
  - Total Run Time = O(number of trees) = O(log N)

Uh…now how would I Insert "1" into this BQ?



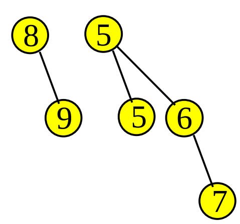


Binomial Queues: Insert

- How would you insert a new item into the queue?
  - Create a single node queue B_0 with new item and Merge with existing queue
  - Again, O(log N) time
- Exercise: Insert 1, 2, 3, …,7 into an empty binomial queue

Binomial Queues: DeleteMin

Insert is easy… how do we DeleteMin?



Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a second queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Run time analysis: How much time do Steps 1 through 4 take for an N-node queue?

Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a second queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Run time analysis: Step 1 is O(log N), steps 2 and 3 are O(1), and step 4 is O(log N). Total time = O(log N)

Next Class:

From Heaps to Hashes

To Do:

Finish Chapter 6 and Start Chapter 5

Homework # 3 has been assigned on the Web
Due Thursday, Feb 13. Start Early!!