Sorting

CSE 326
Data Structures
Unit 15

Reading:
Sections 7.1-7.3 Bubble and Insert sort,
7.5 Heap sort,
Section 3.2.6 Radix sort,
Section 7.6 Mergesort,
Section 7.7 Quicksort,
Section 7.8 Lower bound

Sorting

- Input
  - › an array A of data records
  - › a key value in each data record
  - › a comparison function which imposes a consistent ordering on the keys (e.g., integers)
- Output
  - › reorganize the elements of A such that
    - For any i and j, if i < j then A[i] ≤ A[j]

Consistent Ordering

- The comparison function must provide a consistent *ordering* on the set of possible keys
  - › You can compare any two keys and get back an indication of a < b, a > b, or a = b
  - › The comparison functions must be consistent
    - If **`compare(a,b)`** says a<b, then **`compare(b,a)`** must say b>a
    - If **`compare(a,b)`** says a=b, then **`compare(b,a)`** must say b=a

Why Sort?

- Sorting algorithms are among the most frequently used algorithms in computer science
- Allows binary search of an N-element array in O(log N) time
- Allows O(1) time access to *k*th largest element in the array for any *k*
- Allows easy detection of any duplicates

Evaluating a Sort Algorithm: Time

- How fast is the algorithm?
  - The definition of a sorted array A says that for any $i<j$, $A[i] < A[j]$
  - This means that you need to at least check on each element at the very minimum, I.e., at least O(N)
  - And you could end up checking each element against every other element, which is $O(N^2)$
  - The big question is: How close to O(N) can you get?

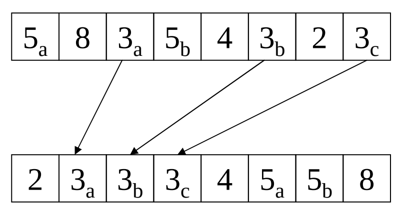
Space

- How much space does the sorting algorithm require in order to sort the collection of items?
  - Is copying needed? O(n) additional space
  - In-place sorting – no copying – O(1) additional space
  - Somewhere in between for “temporary”, e.g. O(logn) space
  - External memory sorting – data so large that does not fit in memory

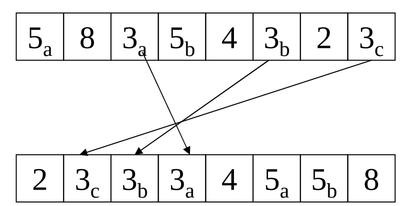
Stability

- Stability: Does it rearrange the order of input data records which have the same key value (duplicates)?
  - E.g. Phone book sorted by name. Now sort by county – is the list still sorted by name within each county?
  - Extremely important property for databases
  - A stable sorting algorithm is one which does not rearrange the order of duplicate keys

Example



Stable Sort



Unstable Sort

Bubble Sort

- "Bubble" elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
  - › Bubble every element towards its correct position
    - last position has the largest element
    - then bubble every element except the last one towards its correct position
    - then repeat until done or until the end of the quarter, whichever comes first ...

Bubblesort

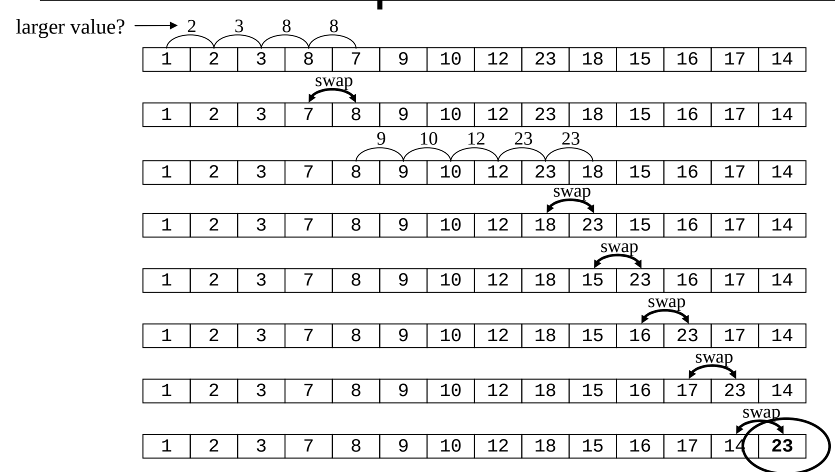
```
bubble(A[1..n]: integer array, n : integer): {
  i, j : integer;
  for i = 1 to n-1 do
    for j = 2 to n–i+1 do
       if A[j-1] > A[j] then SWAP(A[j-1],A[j]);
  }

SWAP(a,b) :  {
  t :integer;
  t:=a; a:=b; b:=t;
}
```

i=1: Largest element is placed at last position
i=k: k^{th} Largest element is placed at k^{th} to last position

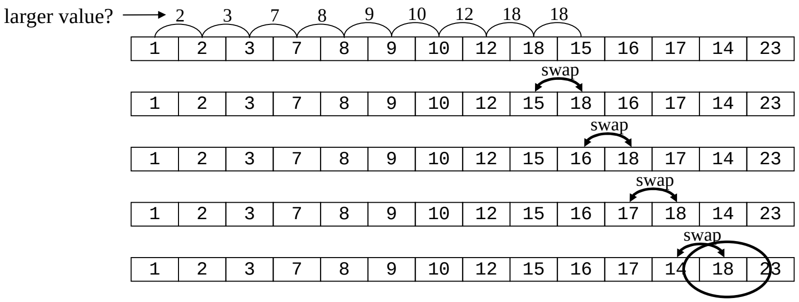
Bubblesort (recursive)

```
bubble(A[1..n]: integer array, n : integer):
  {





  }
```


Put the largest element in its place



Put 2nd largest element in its place



Two elements done, only n-2 more to go ...

Bubble Sort: **Just Say No**

- "Bubble" elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
- We bubblize for i=1 to n (i.e, n times)
- Each bubblization is a loop that makes n-i comparisons
- This is $O(n^2)$

Insertion Sort

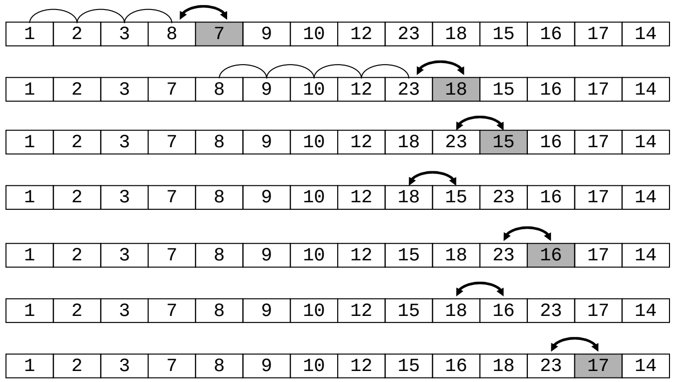
- What if first *k* elements of array are already sorted?
  - 4, 7, 12, 5, 19, 16
- We can shift the tail of the sorted elements list down and then *insert* next element into proper position and we get k+1 sorted elements
  - 4, 5, 7, 12, 19, 16

Insertion Sort

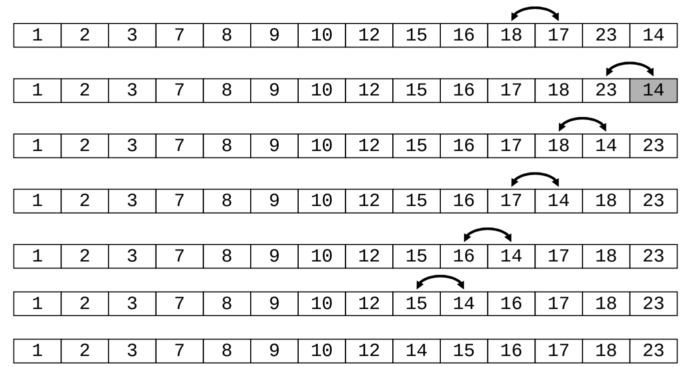
```
InsertionSort(A[1..N]: integer array, N: integer) {
  i, j, temp: integer ;
  for i = 2 to N {
    temp := A[i];
    j := i-1;
    while j > 1 and A[j-1] > temp {
          A[j] := A[j-1]; j := j–1;
          A[j] = temp;
    }
  }
}
```

- Is Insertion sort in place? Stable? Running time = ?
- Have we used something similar before?

Example



Example



Insertion Sort Characteristics

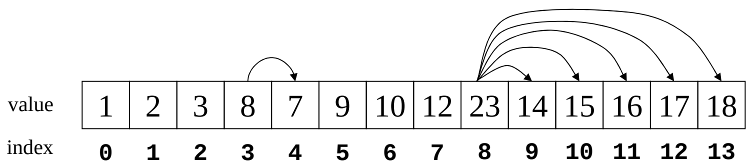
- In place and Stable
- Running time
  - › Worst case is $O(N^2)$
    - reverse order input
    - must copy every element every time
- Good sorting algorithm for almost sorted data
  - › Each item is close to where it belongs in sorted order.

Inversions

- An inversion is a pair of elements in wrong order
  - › $i < j$ but $A[i] > A[j]$
- By definition, a sorted array has no inversions
- So you can think of sorting as the process of removing inversions in the order of the elements

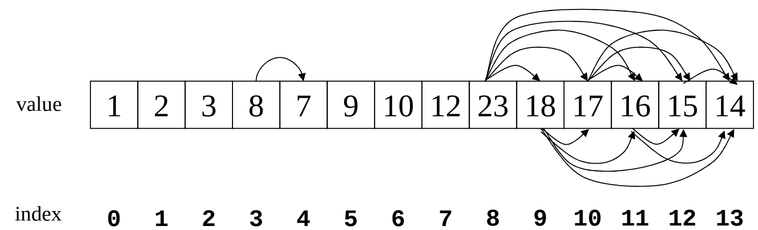
Inversions

- A single value out of place can cause several inversions



Reverse order

- All values out of place (reverse order) causes numerous inversions



Inversions

- Our simple sorting algorithms so far swap adjacent elements and remove just one inversion at a time
  - › Their running time is proportional to number of inversions in array
- Given N distinct keys, the maximum possible number of inversions is

$$(n-1)+(n-2)+\ldots+1=\sum_{i=1}^{n-1} i=\frac{(n-1)n}{2}$$

Inversions and Adjacent Swap Sorts

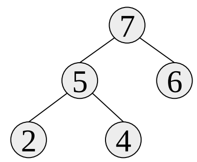
- "Average" list will contain half the max number of inversions = $\frac{(n-1)n}{4}$
  - › So the average running time of Insertion sort is $\Theta(N^2)$
- Any sorting algorithm that only swaps adjacent elements requires $\Omega(N^2)$ time because each swap removes only one inversion (lower bound)

Heap Sort

- We use a Max-Heap
- Root node = A[1]
- Children of A[i] = A[2i], A[2i+1]
- Keep track of current size N (number of nodes)

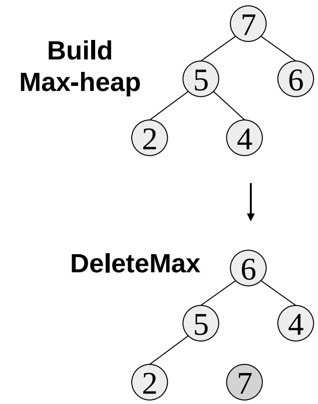
value	7	5	6	2	4			
index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

N = 5



Using Binary Heaps for Sorting

- Build a <u>max-heap</u>
- Do N <u>DeleteMax</u> operations and store each Max element as it comes out of the heap
- Data comes out in largest to smallest order
- Where can we put the elements as they are removed from the heap?

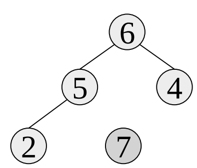


1 Removal = 1 Addition

- Every time we do a DeleteMax, the heap gets smaller by one node, and we have one more node to store
  - › Store the data at the end of the heap array
  - › Not "in the heap" but it is in the heap array

value	6	5	4	2	7			
index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

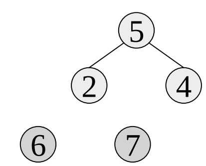
N = 4



Repeated DeleteMax

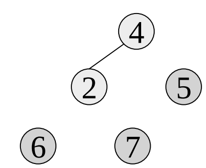
5	2	4	6	7			
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

N = 3



4	2	5	6	7			
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

N = 2

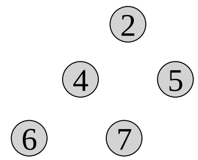


Heap Sort is In-place

- After all the DeleteMaxs, the heap is gone but the array is full and is in sorted order

value	2	4	5	6	7			
index	1	2	3	4	5	6	7	8

N = 0



Heapsort: Analysis

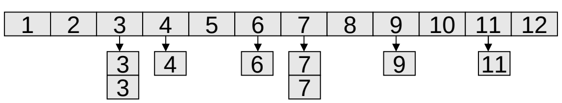
- Running time
  - › time to build max-heap is O(N)
  - › time for N DeleteMax operations is N O(log N)
  - › total time is **O(N log N)**
- Can also show that running time is Ω(N log N) for some inputs,
  - › so *worst case* is Θ**(N log N)**
  - › *Average case* running time is also O(N log N)
- Heapsort is in-place but not stable (why?)

Bucket Sort: Sorting Integers

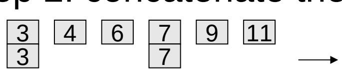
- The goal: sort N numbers, all between 1 to k.
- Example: sort 8 numbers 3,6,7,4,11,3,5,7. All between 1 to 12.
- The method: Use an array of k queues. Queue j (for 1 ≤ j ≤ k) keeps the input numbers whose value is j.
- Each queue is denoted 'a bucket'.
- Scan the list and put the elements in the buckets.
- Output the content of the buckets from 1 to k.

Bucket Sort: Sorting Integers

- Example: sort 8 numbers 3,6,7,4,11,3,9,7 all between 1 to 12.
- Step 1: scan the list and put the elements in the queues



- Step 2: concatenate the queues



3,3,4,6,7,7,9,11

- Time complexity: O(n+k).

Radix Sort: Sorting integers

- Historically goes back to the 1890 census.
- Radix sort = multi-pass bucket sort of integers in the range 0 to B^P-1
- Bucket-sort from least significant to most significant "digit" (base B)
- Requires P(B+N) operations where P is the number of passes (the number of base B digits in the largest possible input number).
- If P and B are constants then O(N) time to sort!

Radix Sort Example

Input data

478
537
9
721
3
38
123
67

Bucket sort
by 1's digit

0	1	2	3	4	5	6	7	8	9
	721		3 123				537 67	478 38	9

After 1st pass

721
3
123
537
67
478
38
9

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

Radix Sort Example

After 1st pass

721
3
123
537
67
478
38
9

Bucket sort
by 10's
digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

After 2nd pass

3
9
721
123
537
38
67
478

Radix Sort Example

After 2nd pass

3
9
721
123
537
38
67
478

Bucket sort
by 100's
digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

After 3rd pass

3
9
38
67
123
478
537
721

Invariant: after k passes the low order k digits are sorted.

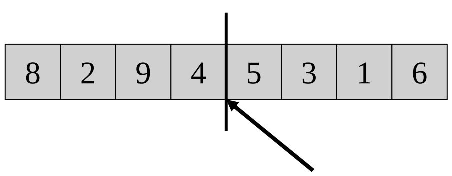
Properties of Radix Sort

- Not in-place
  - › needs lots of auxiliary storage.
- Stable
  - › equal keys always end up in same bucket in the same order.
- Fast
  - › Time to sort N numbers in the range 0 to B^P-1 is O(P(B+N)) (P iterations, B buckets in each)

"Divide and Conquer"

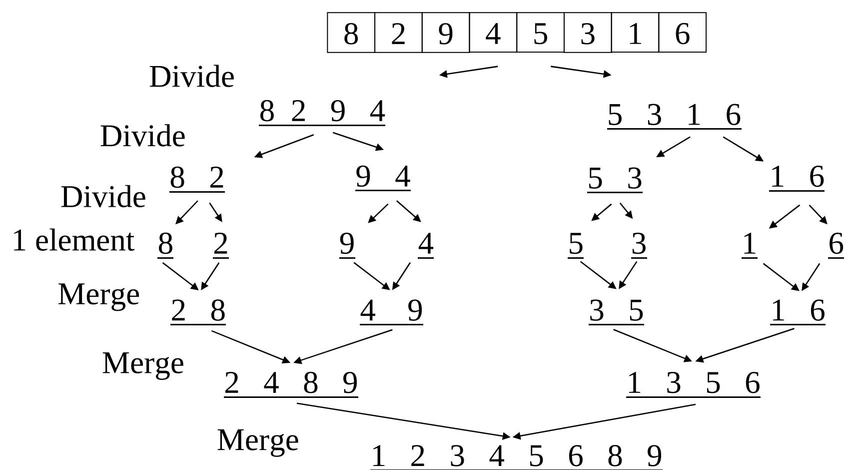
- Very important strategy in computer science:
  - › Divide problem into smaller parts
  - › Independently solve the parts
  - › Combine these solutions to get overall solution
- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves à Mergesort
- **Idea 2 :** Partition array into items that are "small" and items that are "large", then recursively sort the two sets à Quicksort

Mergesort



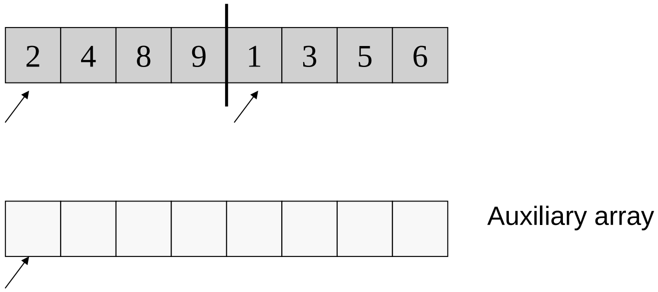
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halves together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



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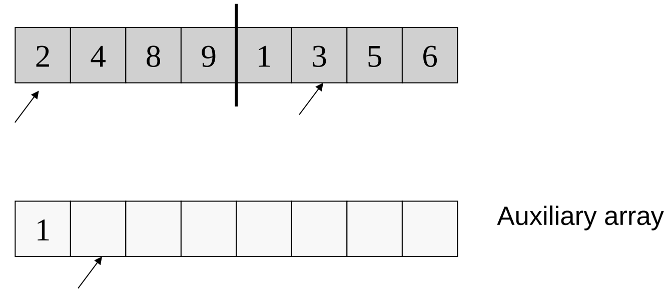

Auxiliary Array

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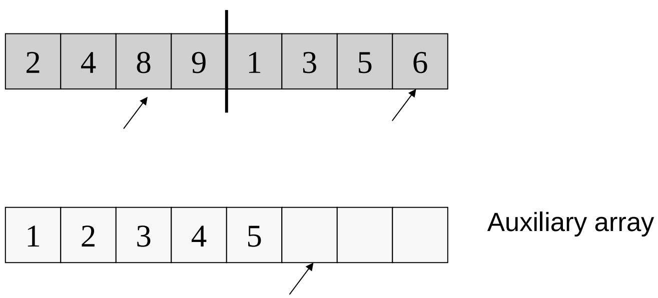
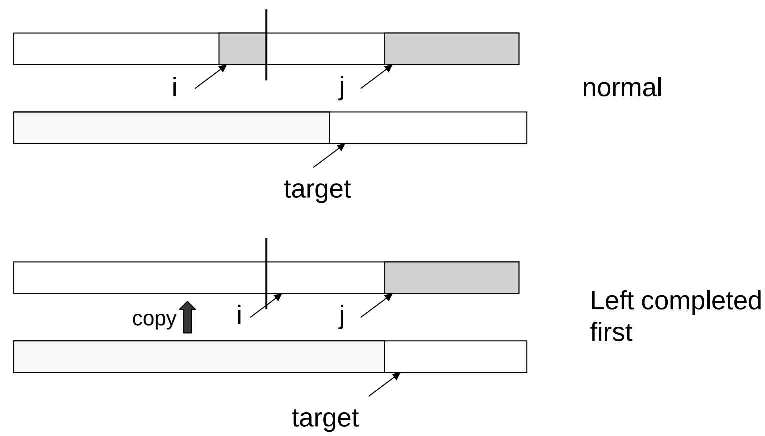
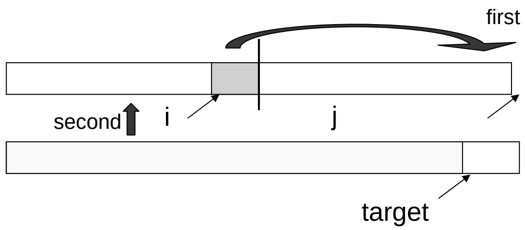


Merging



Merging



Right completed first

Merging

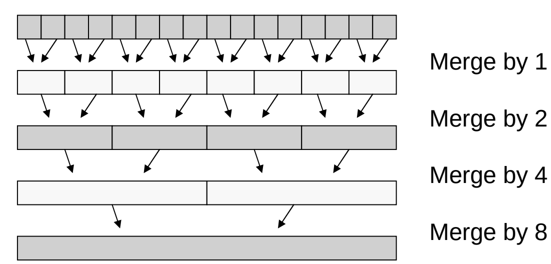
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```


Recursive Mergesort

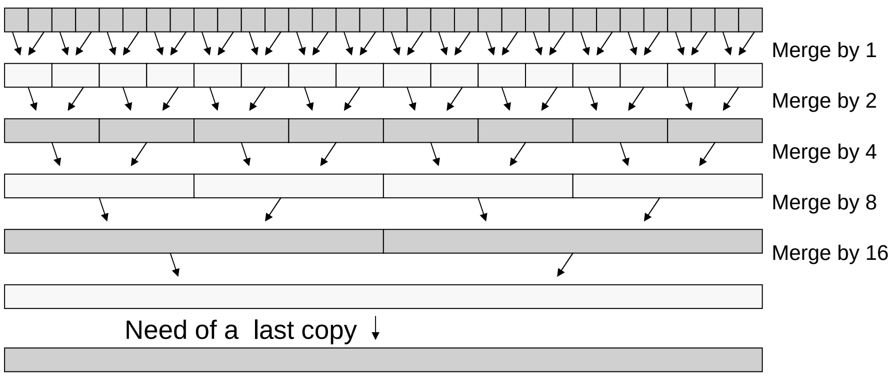
```
Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}
```


Iterative Mergesort



Iterative Mergesort



Iterative Mergesort

```
IterativeMergesort(A[1..n]: integer array, n : integer) : {
//precondition: n is a power of 2//
  i, m, parity : integer;
  T[1..n]: integer array;
  m := 2; parity := 0;
  while m ≤ n do
    for i = 1 to n – m + 1 by m do
       if parity = 0 then Merge(A,T,i,i+m-1);
         else Merge(T,A,i,i+m-1);
    parity := 1 – parity;
    m := 2*m;
  if parity = 1 then
    for i = 1 to n do A[i] := T[i];
}
```

How do you handle non-powers of 2?
How can the final copy be avoided?

Mergesort Analysis

- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

- The recurrence relation for T(N) is:
  - › $T(1) \leq a$
    - base case: 1 element array à constant time
  - › $T(N) \leq 2T(N/2) + dN$
    - Sorting N elements takes
      - – the time to sort the left half
      - – plus the time to sort the right half
      - – plus an O(N) time to merge the two halves
- T(N)= ?

Mergesort Analysis Upper Bound

$$
\begin{aligned}
T(n) &\le 2T(n/2) + dn \qquad \text{Assuming n is a power of 2} \\
&\le 2(2T(n/4) + dn/2) + dn \\
&= 4T(n/4) + 2dn \\
&\le 4(2T(n/8) + dn/4) + 2dn \\
&= 8T(n/8) + 3dn \\
&\vdots \\
&\le 2^k T(n/2^k) + kdn \\
&= nT(1) + kdn \qquad \text{if } n = 2^k \qquad n = 2^k,\ k = \log n \\
&\le cn + dn \log_2 n \\
&= O(n \log n)
\end{aligned}
$$

Properties of Mergesort

- Not in-place
  - › Requires an auxiliary array (O(n) extra space)
- Stable
  - › Make sure that left is sent to target on equal values.
- Iterative Mergesort reduces copying.

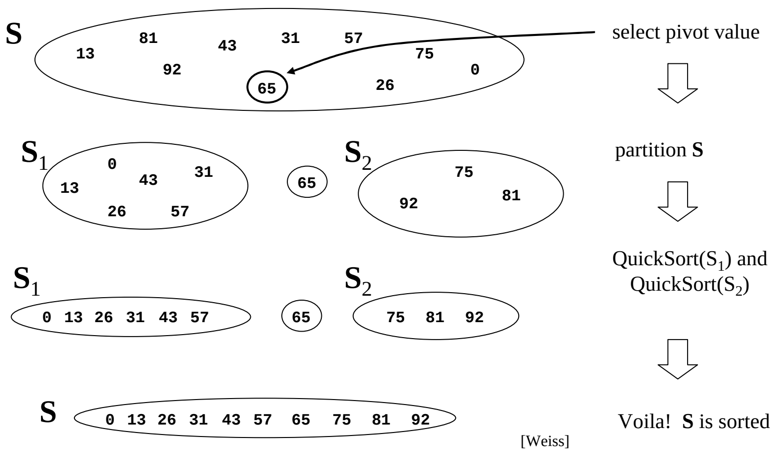
Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - › Partition array into left and right sub-arrays
    - Choose an element of the array, called pivot
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - › Recursively sort left and right sub-arrays
  - › Concatenate left and right sub-arrays in O(1) time

"Four easy steps"

- To sort an array **S**
  1. If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  2. Pick an element *v* in **S**. This is the *pivot* value.
  3. Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \le v$}, and $\mathbf{S}_2$ = {all values $x \ge v$}.
  4. Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

The steps of QuickSort



Details, details

- Implementing the actual partitioning
- Picking the pivot
  - › want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible
- Dealing with cases where an element equals the pivot

Quicksort Partitioning

- Need to partition the array into left and right sub-arrays
  - › the elements in left sub-array are ≤ pivot
  - › elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - › Choose an element from the array as the pivot
  - › Make one pass through the rest of the array and swap as needed to put elements in partitions

Partitioning:Choosing the pivot

- One implementation (there are others)
  - › median3 finds pivot and sorts left, center, right
    - Median3 takes the median of leftmost, middle, and rightmost elements
    - An alternative is to choose the pivot randomly (need a random number generator; “expensive”)
    - Another alternative is to choose the first element (but can be very bad. Why?)
  - › Swap pivot with next to last element

Partitioning in-place

- Set pointers i and j to start and end of array
- Increment i until you hit element A[i] > pivot
- Decrement j until you hit element A[j] < pivot
- Swap A[i] and A[j]
- Repeat until i and j cross
- Swap pivot (at A[N-2]) with A[i]

Example

Choose the pivot as the median of three

0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

Median of 0, 6, 8 is 6. Pivot is 6

0	1	4	9	7	3	5	2	6	8
i								j	

Place the largest at the right
and the smallest at the left.
Swap pivot with next to last element.

Example

i								j	
0	1	4	9	7	3	5	2	6	8
			i					j	
0	1	4	**9**	7	3	5	2	6	8
			i				j		
0	1	4	9	7	3	5	**2**	6	8
			i				j		
0	1	4	**2**	7	3	5	**9**	6	8

Move i to the right up to A[i] larger than pivot.
Move j to the left up to A[j] smaller than pivot.
Swap

Example

				i			j		
0	1	4	2	**7**	3	5	9	6	8
				i		j			
0	1	4	2	7	3	**5**	9	6	8
				i		j			
0	1	4	2	**5**	3	**7**	9	6	8
						i j			
0	1	4	2	5	3	**7**	9	6	8
					j	i			
0	1	4	2	5	**3**	7	9	6	8
					j	i			
0	1	4	2	5	3	**6**	9	**7**	8

Cross-over i > j

S_1 < pivot   pivot S_2 > pivot

Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don't use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half at each recursion
  - › T(0) = T(1) = O(1)
    - constant time if 0 or 1 element
  - › For N > 1, 2 recursive calls plus linear time for partitioning
  - › T(N) = 2T(N/2) + O(N)
    - Same recurrence relation as Mergesort
  - › T(N) = <u>O(N log N)</u>

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - › $T(N) \le a$ for $N \le C$
  - › $T(N) \le T(N-1) + bN$
  - › $\le T(N-2) + b(N-1) + bN$
  - › $\le T(C) + b(C+1) + \dots + bN$
  - › $\le a + b(C + (C+1) + (C+2) + \dots + N)$
  - › $T(N) = O(N^2)$
- Fortunately, *average case performance* is O(N log N) (see text for proof)

Properties of Quicksort

- Not stable because of long distance swapping.
- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- "In-place", but uses auxiliary storage because of recursive call (O(logn) space).
- O(n log n) average case performance, but $O(n^2)$ worst case performance.

How fast can we sort?

- Heapsort, Mergesort, and Quicksort all run in O(N log N) best case running time
- Can we do any better?
- No, if sorting is comparison-based.
- We saw that radix sort is O(N) but it is only for integers from bounded-range.

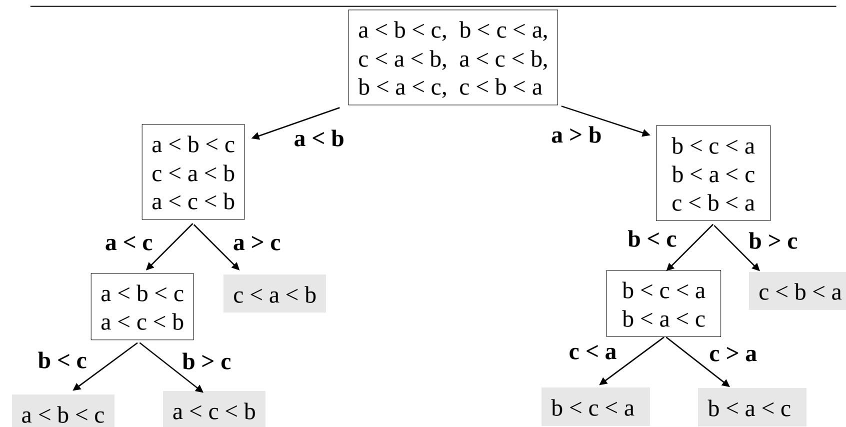
Sorting Model

- Recall the basic assumption: we can only compare two elements at a time
  - › we can only reduce the possible solution space by half each time we make a comparison
- Suppose you are given N elements
  - › Assume no duplicates
- How many possible orderings can you get?
  - › Example: a, b, c (N = 3)

Permutations

- How many possible orderings can you get?
  - › Example: a, b, c (N = 3)
  - › (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  - › 6 orderings = 3•2•1 = 3! (i.e., “3 factorial”)
  - › All the possible permutations of a set of 3 elements
- For N elements
  - › N choices for the first position, (N-1) choices for the second position, …, (2) choices, 1 choice
  - › N(N-1)(N-2)⋯(2)(1)= N! possible orderings

Decision Tree



The leaves contain all the possible orderings of a, b, c

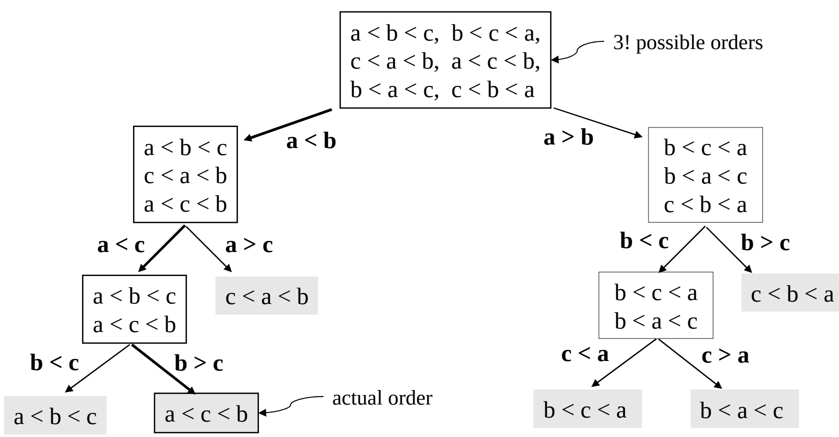
Decision Trees

- A Decision Tree is a Binary Tree such that:
  - › Each node = a set of orderings
    - i.e., the remaining solution space
  - › Each edge = 1 comparison
  - › Each leaf = 1 unique ordering
  - › How many leaves for N distinct elements?
    - N!, i.e., a leaf for each possible ordering
- Only 1 leaf has the ordering that is the desired correctly sorted arrangement

Decision Trees and Sorting

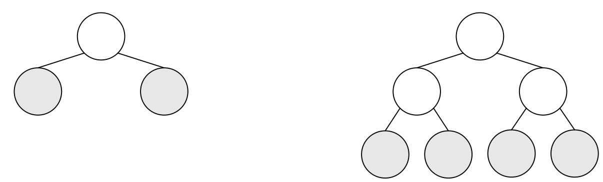
- Every comparison-based sorting algorithm corresponds to a decision tree
  - › Finds correct leaf by choosing edges to follow
    - i.e., by making comparisons
  - › Each decision reduces the possible solution space by one half
- Run time is ≥ maximum no. of comparisons
  - › maximum number of comparisons is the length of the longest path in the decision tree, i.e. the height of the tree

Decision Tree Example



How many leaves on a tree?

- Suppose you have a binary tree of height d . How many leaves can the tree have?
  - › d = 1 à at most 2 leaves,
  - › d = 2 à at most 4 leaves, etc.



Lower bound on Height

- A binary tree of height d has at most **2^d** leaves
  - › depth d = 1 à 2 leaves, d = 2 à 4 leaves, etc.
  - › Can prove by induction
- Number of leaves, $L \leq 2^d$
- Height $d \geq \log_2 L$
- The decision tree has N! leaves
- So the decision tree has height $d \geq \log_2(N!)$

log(*N*!) is Ω(*N*log*N*)

$$\log(N!) = \log\big(N \cdot (N-1) \cdot (N-2) \cdots (2) \cdot (1)\big)$$

$$= \log N + \log(N-1) + \log(N-2) + \cdots + \log 2 + \log 1$$

select just the first N/2 terms

$$\geq \log N + \log(N-1) + \log(N-2) + \cdots + \log \frac{N}{2}$$

each of the selected terms is ≥ logN/2

$$\geq \frac{N}{2} \log \frac{N}{2}$$

$$\geq \frac{N}{2}(\log N - \log 2) = \frac{N}{2}\log N - \frac{N}{2}$$

$$= \Omega(N \log N)$$

$$n! \approx \sqrt{2\pi n}(n/e)^n$$

Sterling's formula

Summary of Sorting

- Sorting choices:
  - › $O(N^2)$ – Bubblesort, Insertion Sort
  - › O(N log N) average case running time:
    - Heapsort: In-place, not stable.
    - Mergesort: O(N) extra space, stable.
    - Quicksort: claimed fastest in practice but, $O(N^2)$ worst case. Needs extra storage for recursion. Not stable.
  - › Run time of any comparison-based sorting algorithm is Ω(N log N)
  - › O(N) – Radix Sort: fast and stable. Not comparison based. Not in-place.