Binary Heaps

CSE 326
Data Structures
Lecture 11

Readings and References

- Reading
  - Sections 6.1-6.4

A New Problem…

- Application: Find the smallest (or highest priority) item quickly
  - Operating system needs to schedule jobs according to priority
  - Doctors in ER take patients according to severity of injuries
  - Event simulation (bank customers arriving and departing, ordered according to when the event happened)

Priority Queue ADT

- Priority Queue can efficiently do:
  - FindMin (and DeleteMin)
  - Insert
- What if we use…
  - Lists: If sorted, what is the run time for Insert and FindMin? Unsorted?
  - Binary Search Trees: What is the run time for Insert and FindMin?

Less flexibility → More speed

- Lists
  - If sorted: FindMin is O(1) but Insert is O(N)
  - If not sorted: Insert is O(1) but FindMin is O(N)
- Balanced Binary Search Trees (BSTs)
  - Insert is O(log N) and FindMin is O(log N)
- BSTs look good but…
  - BSTs are efficient for all Finds, not just FindMin
  - We only need FindMin

Better than a speeding BST

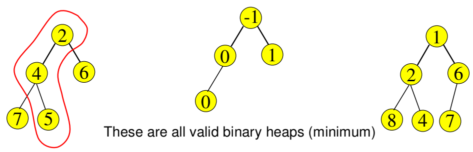
- We can do better than Balanced Binary Search Trees?
  - Very limited requirements: Insert, FindMin, DeleteMin
  - FindMin is O(1)
  - Insert is O(log N)
  - DeleteMin is O(log N)

Binary Heaps

- A binary heap is a binary tree that is:
  - Complete: the tree is completely filled except possibly the bottom level, which is filled from left to right
  - Satisfies the heap order property
    - every node is less than or equal to its children
    - or every node is greater than or equal to its children
- The root node is always the smallest node
  - or the largest, depending on the heap order

Heap order property

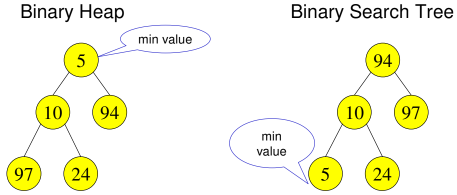
- A heap provides limited ordering information
- Each *path* is sorted, but the subtrees are not sorted relative to each other
  - A binary heap is NOT a binary search tree



These are all valid binary heaps (minimum)

Binary Heap vs Binary Search Tree

Binary Heap | Binary Search Tree

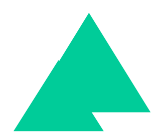


Parent is less than both left and right children

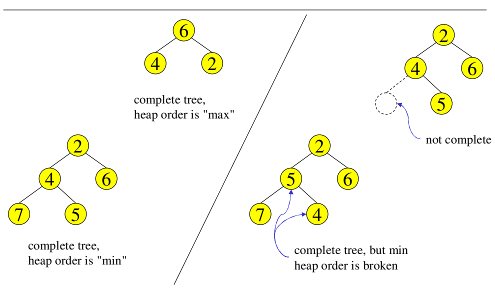
Parent is greater than left child, less than right child

Structure property

- A binary heap is a complete tree
  - All nodes are in use except for possibly the right end of the bottom row

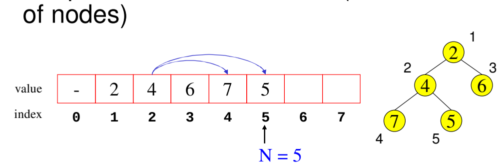


Examples



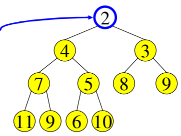
Array Implementation of Heaps (Implicit Pointers)

- Root node = A[1]
- Children of A[i] = A[2i], A[2i + 1]
- Keep track of current size N (number of nodes)



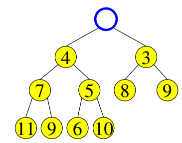
FindMin and DeleteMin

- FindMin: Easy!
  - › Return root value A[1]
  - › Run time = ?
- DeleteMin:
  - › Delete (and return) value at root node



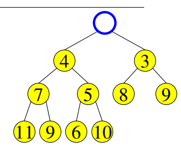
DeleteMin

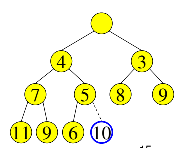
- Delete (and return) value at root node



Maintain the Structure Property

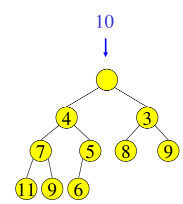
- We now have a “Hole” at the root
  - › Need to fill the hole with another value
- When we get done, the tree will have one less node and must still be complete



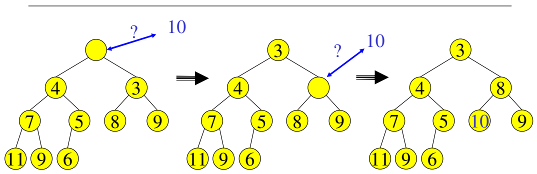


Maintain the Heap Property

- The last value has lost its node
  - › we need to find a new place for it
- We can do a simple insertion sort operation to find the correct place for it in the tree



DeleteMin: Percolate Down



- Keep comparing with children A[2i] and A[2i + 1]
- Copy smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- What is the run time?

Percolate Down

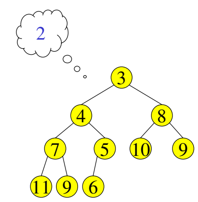
```
PercDown(i:integer, x :integer): {
// N is the number of entries in queue//
j : integer;
Case{
  2i > N : A[i] := x; //at bottom//
  2i = N : if A[2i] < x then
              A[i] := A[2i]; A[2i] := x;
           else A[i] := x;
  2i < N : if A[2i] < A[2i+1] then j := 2i;
           else j := 2i+1;
           if A[j] < x then
              A[i] := A[j]; PercDown(j,x);
           else A[i] := x;
}}
```

DeleteMin: Run Time Analysis

- Run time is O(depth of heap)
- A heap is a complete binary tree
- Depth of a complete binary tree of N nodes?
  - height = $\lceil \log_2(N) \rceil - 1$
- Run time of DeleteMin is O(log N)

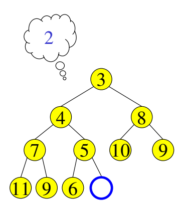
Insert

- Add a value to the tree
- Structure and heap order properties must still be correct when we are done



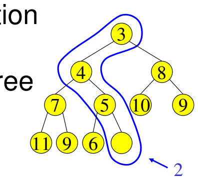
Maintain the Structure Property

- The only valid place for a new node in a complete tree is at the end of the array
- We need to decide on the correct value for the new node, and adjust the heap accordingly

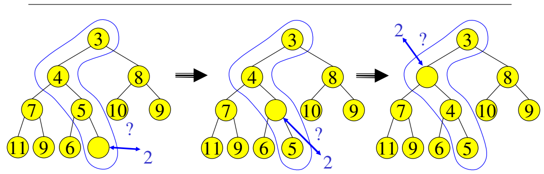


Maintain the Heap Property

- The new value goes where?
- We can do a simple insertion sort operation to find the correct place for it in the tree

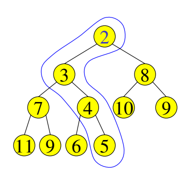


Insert: Percolate Up



- Start at last node and keep comparing with parent A[i/2]
- If parent larger, copy parent down and go up one level
- Done if parent ≤ item or reached top node A[1]
- Run time?

Insert: Done



- Run time?

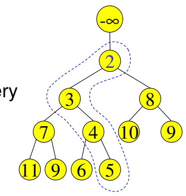
PercUp

- Class participation
- Define PercUp which percolates new entry to correct spot.
- Note: the parent of i is i/2

```
PercUp(i : integer, x : integer): {
????
}
```


Sentinel Values

- Every iteration of Insert needs to test:
  - if it has reached the top node A[1]
  - if parent ≤ item
- Can avoid first test if A[0] contains a very large negative value
  - sentinel -∞ < item, for all items
- Second test alone always stops at top



value	-∞	2	3	8	7	4	10	9	11	9	6	5		
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13

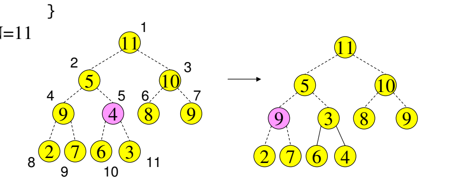
Binary Heap Analysis

- Space needed for heap of N nodes: O(MaxN)
  - An array of size MaxN, plus a variable to store the size N, plus an array slot to hold the sentinel
- Time
  - FindMin: O(1)
  - DeleteMin and Insert: O(log N)
  - BuildHeap from N inputs : O(N)

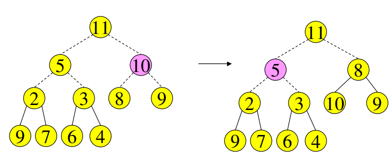
Build Heap

```
BuildHeap {
for i = N/2 to 1 by –1 PercDown(i,A[i])
}
```

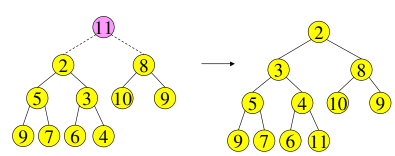
N=11



Build Heap



Build Heap



Analysis of Build Heap

- Assume $N = 2^K - 1$
  - Level 1: k -1 steps for 1 item
  - Level 2: k - 2 steps of 2 items
  - Level 3: k - 3 steps for 4 items
  - Level i : k - i steps for 2^{i-1} items

$$\text{Total Steps} = \sum_{i=1}^{k-1} (k-i)2^{i-1} = 2^k - k - 1$$
$$= O(N)$$

Other Heap Operations

- Find(X, H): Find the element X in heap H of N elements
  - What is the running time? O(N)
- FindMax(H): Find the maximum element in H
  - What is the running time? O(N)
- We sacrificed performance of these operations in order to get O(1) performance for FindMin

Other Heap Operations

- DecreaseKey(P,Δ,H): Decrease the key value of node at position P by a positive amount Δ. eg, to increase priority
  - First, subtract Δ from current value at P
  - Heap order property may be violated
  - so percolate up to fix
  - Running Time: O(log N)

Other Heap Operations

- IncreaseKey(P,Δ,H): Increase the key value of node at position P by a positive amount Δ. eg, to decrease priority
  - First, add Δ to current value at P
  - Heap order property may be violated
  - so percolate down to fix
  - Running Time: O(log N)

Other Heap Operations

- Delete(P,H): E.g. Delete a job waiting in queue that has been preemptively terminated by user
  - Use DecreaseKey(P,∞,H) followed by DeleteMin
  - Running Time: O(log N)

Other Heap Operations

- Merge(H1,H2): Merge two heaps H1 and H2 of size O(N). H1 and H2 are stored in two arrays.
  - Can do O(N) Insert operations: O(N log N) time
  - Better: Copy H2 at the end of H1 and use BuildHeap. Running Time: O(N)

PercUp Solution

```
PercUp(i : integer, x : integer): {
if i = 1 then A[1] := x
else if A[i/2] < x then
       A[i] := x;
     else
       A[i] := A[i/2];
       Percup(i/2,x);
}
```