Loops and invariants

CSE 331
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Reasoning about loops

A loop represents an unknown number of paths

- Case analysis is problematic
- Recursion presents the same problem as loops

Cannot enumerate all paths

- This is what makes testing and reasoning hard

Things to prove about a loop:

1. It computes the correct value
2. It terminates (no infinite loop)

Reasoning about loops: values and termination

// assert x ≥ 0 & y = 0

```
while (x != y) {
    y = y + 1;
}
```

// assert x = y

Does "x=y" hold after this loop?

Does this loop terminate?

1) Pre-assertion guarantees that $x \geq y$

2) Every time through loop

   $x \geq y$ holds at the test – and if body is entered, $x > y$

   y is incremented by 1

   x is unchanged

   Therefore, y is closer to x (but $x \geq y$ still holds)

3) Since there are only a finite number of integers between x and y, y will eventually equal x

4) Execution exits the loop as soon as $x = y$ (but $x \geq y$ still holds)

Understanding loops by induction

- We just made an inductive argument
  - Inducting over the *number of iterations*
- Computation induction
  - Show that conjecture holds if zero iterations
  - Show that it holds after n+1 iterations
    - (assuming that it holds after n iterations)
- Two things to prove
  1. Some property is preserved (known as "partial correctness"), *if* the code terminates
     - Loop invariant is preserved by each iteration, if the iteration completes
  2. The loop completes (known as "termination")
     - The "decrementing function" is reduced by each iteration and cannot be reduced forever

How to choose a loop invariant, LI

{ P }
```
while (b) S;
```
{ Q }

Find an invariant, LI, such that

1. $P \Rightarrow LI$ // true initially
2. { LI & b } **S** { LI } // true if the loop executes once
3. $(LI \,\&\, \neg b) \Rightarrow Q$ // establishes the postcondition

Finding the invariant is the key to reasoning about loops

Fact: For any loop and pre/post conditions it satisfies, there exists a loop invariant sufficient to prove them

(Equivalently: inductive assertions is a "complete method of proof")

(We will not yet prove loop termination; it is sufficient to know that if loop terminates, Q will hold.)

Loop invariant for the example

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

A suitable invariant:

$LI = x \geq y$

1. $x \geq 0 \;\&\; y = 0 \Rightarrow LI$ // true initially
2. $\{ LI \;\&\; x \neq y \}$ `y = y+1;` $\{ LI \}$ // true if the loop executes once
3. $(LI \;\&\; \neg(x \neq y)) \Rightarrow x = y$ // establishes the postcondition

Termination via reduction to natural numbers

Total correctness = partial correctness + termination

We have not established that the loop terminates

Suppose that the loop always reduces some variable's value. Does the loop terminate if the variable is a

- Natural number?
- Integer?
- Non-negative real number?
- Boolean?
- ArrayList?

The loop terminates if:

- each variable value maps to a natural number, and
- the number decreases on each iteration

Well-ordered sets

We don’t *have* to use the natural numbers

The loop terminates if the variable values are, or can be mapped to, (a subset of) a well-ordered set

- Ordered set
- Every non-empty subset has a least element

The natural numbers are the best choice

Decrementing function

Decrementing function D(X)

Maps state (program variables) to a natural number

Consider: `while (b) S;`

We seek D(X), where X is the state, such that

1. An execution of the loop reduces the function's value:
   $\{ LI \ \& \ b \} \ \mathbf{S} \ \{ D(X_{post}) < D(X_{pre}) \}$
2. If the function's value is minimal, the loop terminates:
   $(LI \ \& \ D(X) = 0) \Rightarrow \neg b$

Proving termination

// assert $x \geq 0$ & $y = 0$
// Loop invariant: $x \geq y$
// Loop decrements: (x-y)

```
while (x != y) {
    y = y + 1;
}
```

// assert $x = y$

Is this a good decrementing function?

1. Does the loop reduce the decrementing function's value?

   // assert ($y \neq x$); let $d_{pre} = (x-y)$

   ```
   y = y + 1;
   ```

   // assert $(x_{post} – y_{post}) < d_{pre}$

2. If the function has minimum value, does the loop exit?

   $(x \geq y \;\&\; x - y = 0) \Rightarrow (x = y)$

Choosing loop invariants

For non-looping code, the wp (weakest precondition) function enables proofs

For loops, you have to guess:

- The loop invariant
- The decrementing function

Then, use reasoning techniques to prove the goal property

If the proof doesn't work:

- Maybe you chose a bad invariant or decrementing function
  - Choose another and try again
- Maybe the loop is incorrect
  - Fix the code

Automatically choosing loop invariants is a research topic

When to use code proofs for loops

Most of your loops need no proofs

```
for (String name : friends) { ... }
```

When you are unsure about a loop,
write loop invariants and decrementing functions

If a loop is not working:

- Add invariant and decrementing function if missing
- Write code to check them
- Understand why the code doesn't work
- Reason to ensure that no similar bugs remain

Example: Factorial

{ n≥0 ∧ t=n }

```
r=1;
```

n≥0 ∧ t=n ∧ r=1

```
while (n≠0) {
```

r=t!/n! ∧ t≥n≥0

```
  r=r*n;
```

r=t!/(n-1)! ∧ t≥n>0

```
  n=n-1;
```

r=t!/n! ∧ t≥n≥0

```
}
```

{ r=t! }

Example: Quotient and remainder

```
r := x;
q := 0;
while (y <= r) {
  r := r - y;
  q := 1 + q;
}
```

$x = x + y \times 0$

$x = r + y \times 0$

$x = r + y \times q$

$\{ x = r + y \times q \text{ and } y > r \}$

Example: Greatest common divisor

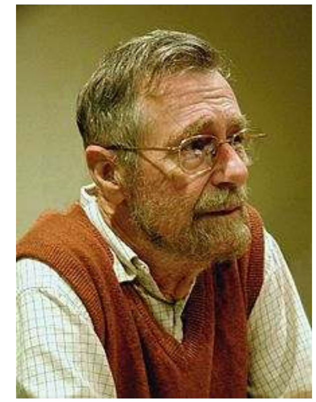
```
{ x1 > 0  ∧  x2 > 0 }
y1:=x1;
y2:=x2;
while (y1≠y2) do
   if  y1>y2
      then y1 := y1-y2
      else y2 := y2-y1
{ y1 = gcd(x1,x2) }
```

Recall: if y1,y2 are both positive integers, then:

- If y1>y2 then gcd(y1,y2)=gcd(y1-y2,y2)
- If y2>y1 then gcd(y1,y2)=gcd(y1,y2-y1)
- If y1-y2 then gcd(y1,y2)=y1=y2

Invariants help you write code as well as prove it correct

Dutch National Flag problem:
Given an array containing balls of three colors, arrange them with like colors together and in the right order



Edsgar Dijkstra

- Precondition:
- Postcondition:
- Loop invariant: