Understanding ADTs

CSE 331
University of Washington

Michael Ernst

Ways to get your design right

The hard way
- Start hacking
- When something doesn't work, hack some more
  - How do you know it doesn't work?
  - Need to reproduce the errors your users experience
- Apply caffeine liberally

The easier way
- Plan first (specs, system decomposition, tests, ...)
- Less apparent progress upfront
- Faster completion times
- Better delivered product
- Less frustration

Ways to verify your code

The hard way: hacking
- Make up some inputs
- If it doesn't crash, ship it
- When it fails in the field, attempt to debug

An easier way: systematic testing
- Reason about possible behaviors and desired outcomes
- Construct simple tests that exercise all behaviors

Another way that can be easy: reasoning
- Prove that the system does what you want
  - Rep invariants are preserved
  - Implementation satisfies specification
- Proof can be formal or informal (we will be informal)
- Complementary to testing

Uses of reasoning

Goal: correct code

- Verify that rep invariant is satisfied
- Verify that the implementation satisfies the spec
- Verify that client code behaves correctly

  Assuming that the implementation is correct

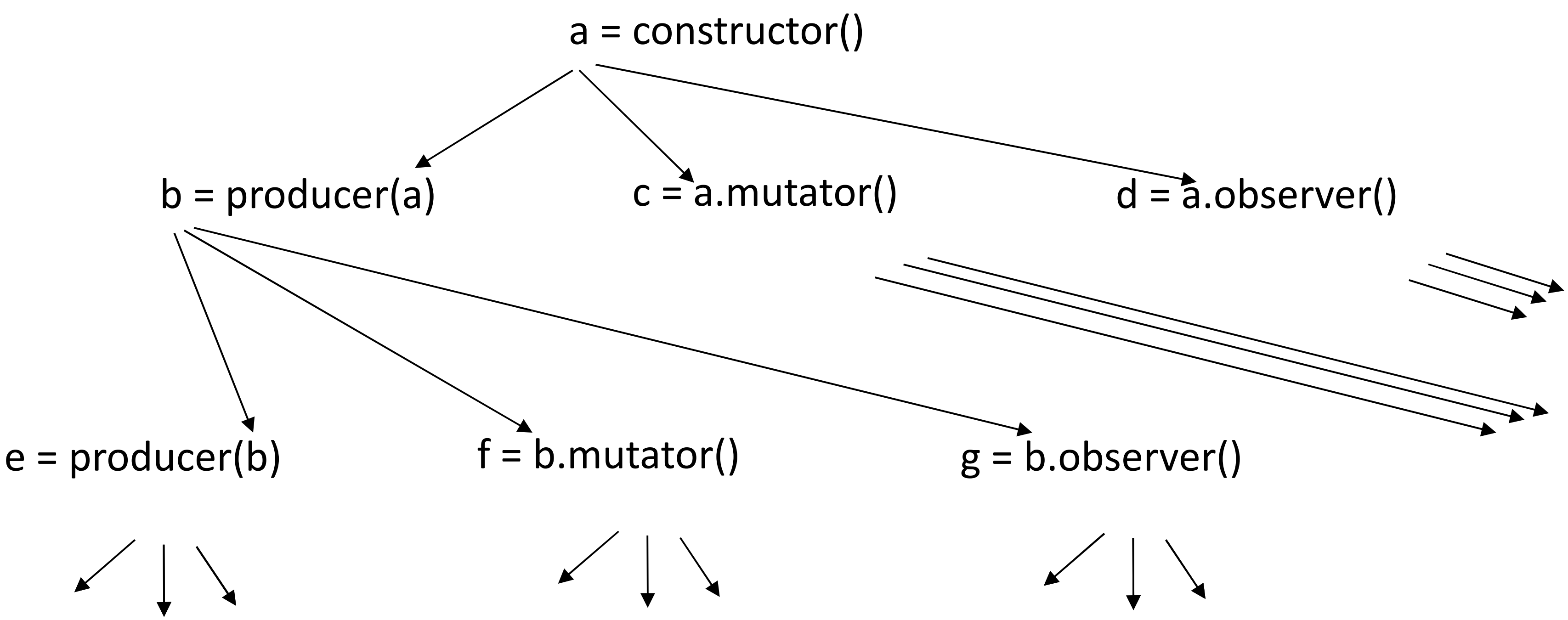
Goal: Demonstrate that rep invariant is satisfied

- Exhaustive testing
  - Create every possible object of the type
  - Check rep invariant for each object
  - Problem: impractical
- Limited testing
  - Choose representative objects of the type
  - Check rep invariant for each object
  - Problem: did you choose well?
- Reasoning
  - Prove that all objects of the type satisfy the rep invariant
  - Sometimes easier than testing, sometimes harder
  - Every good programmer uses it as appropriate

All possible objects (and values) of a type

- Make a new object
  - constructors
  - producers
- Modify an existing object
  - mutators
  - observers, producers (why?)
- Limited number of operations, but infinitely many objects
  - Maybe infinitely many values as well

Examples of making objects



Infinitely many possibilities

We cannot perform a proof that considers each possibility case-by-case

Solution: induction

Induction: technique for proving *infinitely* many facts using *finitely* many proof steps

For constructors ("basis step")

- Prove the property holds on exit

For all other methods ("inductive step")

- Prove that:
  - if the property holds on entry, then it holds on exit

If the basis and inductive steps are true:

- There is no way to make an object for which the property does not hold
- Therefore, the property holds for all objects

A counter class

```
// spec field: count
// abstract invariant:  count ≥ 0
class Counter {
  // counts up starting from 0
  Counter();
  // returns a copy of this counter
  Counter clone();
  // increments the value that this represents:
  // count_post = count_pre + 1
  void increment();
  // returns count
  BigInteger getValue();
}
```

Is the abstract invariant satisfied by these method specs?

Proof by contradiction: where was the invariant first violated?

Inductive proof

- Base case: invariant is satisfied by constructor
- Inductive case:
  - If invariant is satisfied on entry to clone, then invariant is satisfied on exit
  - If invariant is satisfied on entry to increment, then invariant is satisfied on exit
  - If invariant is satisfied on entry to getValue, then invariant is satisfied on exit
- Conclusion: invariant is always satisfied

Inductive proof that x+1 > x

ADT: the natural numbers (non-negative integers)

- constructor: 0 (zero)
- producer: succ (successor: succ(x) = x+1)
- mutators: none
- observers: value

Axioms:

1. succ(0) > 0
2. (succ(i) > succ(j)) ⇔ i > j

Goal: prove that for all natural numbers x, succ(x) > x

Possibilities for x:

1. x is 0
   - succ(0) > 0 — axiom #1
2. x is succ(y) for some y
   - succ(y) > y — assumption
   - succ(succ(y)) > succ(y) — axiom #2
   - succ(x) > x — def of x = succ(y)

Outline for remainder of lecture

1. Prove that rep invariant is satisfied
2. Prove that client code behaves correctly

   (Assuming that the implementation is correct)

CharSet abstraction

```
// Overview: A CharSet is a finite mutable set of chars.

// effects: creates a fresh, empty CharSet
public CharSet ( )

// modifies: this
// effects: this_post = this_pre U {c}
public void insert (char c);

// modifies: this
// effects: this_post = this_pre - {c}
public void delete (char c);

// returns: (c ∈ this)
public boolean member (char c);

// returns: cardinality of this
public int size ( );
```

Implementation of CharSet

```
// Rep invariant:  elts has no nulls and no duplicates
List<Character> elts;

public CharSet() {
  elts = new ArrayList<Character>();
}
public void delete(char c) {
  elts.remove(new Character (c));
}
public void insert(char c) {
  if (! member(c))
    elts.add(new Character(c));
}
public boolean member(char c) {
  return elts.contains(new Character(c));
}
…
```

Proof of CharSet representation invariant

Rep invariant: elts has no nulls and no duplicates

Base case: constructor

```
public CharSet() {
  elts = new ArrayList<Character>();
}
```

This satisfies the rep invariant

Inductive step:

For each other operation:

Assume rep invariant holds before the operation

Prove rep invariant holds after the operation

Inductive step, `member`

Rep invariant: elts has no nulls and no duplicates

```
public boolean member(char c) {
  return elts.contains(new Character(c));
}
```

`contains` doesn't change **`elts`**, so neither does **`member`**.
Conclusion: rep invariant is preserved.

Why do we even need to check **`member`**?
- After all, the specification says that it does not mutate set.

Reasoning must account for all possible arguments
- It's best not to involve the specific values in the proof

Inductive step, `delete`

Rep invariant: elts has no nulls and no duplicates

```
public void delete(char c) {
  elts.remove(new Character(c));
}
```

`List.remove` has two behaviors:

- leaves **`elts`** unchanged, or
- removes an element.

Rep invariant can only be made false by adding elements.

Conclusion: rep invariant is preserved.

Inductive step, `insert`

Rep invariant: elts has no nulls and no duplicates

```
public void insert(char c) {
  if (! this.member(c))
    elts.add(new Character(c));
}
```

If **c** $\in$ **elts**$_{pre}$:

elts is unchanged $\Rightarrow$ rep invariant is preserved

If **c** $\notin$ **elts**$_{pre}$:

new element is not null or a duplicate $\Rightarrow$ rep invariant is preserved

Reasoning about mutations to the rep

Inductive step must consider all possible changes to the rep

- A possible source of changes: representation exposure
- If the proof does not account for this, then the proof is invalid

An important reason to protect the rep:

- Compiler can help verify that there are no external changes

Induction for reasoning about uses of ADTs

Induction on specification, not on code

Abstract values (e.g., specification fields) may differ from concrete representation

Can ignore observers, since they do not affect abstract state

    How do we know that?

Axioms:

- specs of operations
- axioms of types used in overview parts of specifications

LetterSet (case-insensitive character set)

// A LetterSet is a mutable finite set of characters.
// No LetterSet contains two chars with the same lower-case representation.

// effects: creates an empty LetterSet
public LetterSet ( );

// Insert c if this contains no other char with same lower-case representation.
// modifies: this
// effects: $this_{post}$ = if ($\exists c_1 \in this_{pre}$ s.t. toLowerCase(c_1) = toLowerCase(c))
//   then $this_{pre}$
//   else $this_{pre}$ U {c}
public void insert (char c);

// modifies: this
// effects: $this_{post} = this_{pre} - \{c\}$
public void delete (char c);

// returns: ($c \in$ this)
public boolean member (char c);

// returns: |this|
public int size ( );

Attempt #1

Goal: prove that no LetterSet contains upper- and lower-case versions of a letter

Property P(X) = $\neg\exists c_1, c_2 \in X\ [\text{toLowerCase}(c_1) = \text{toLowerCase}(c_2)]$

Consider an arbitrary LetterSet S

Prove P(S); that is: $\neg\exists c_1, c_2 \in S\ [\text{toLowerCase}(c_1) = \text{toLowerCase}(c_2)]$

How might S have been made?

$\xrightarrow{\text{constructor}} S$ Base case

$T \xrightarrow{\text{T.insert(c)}} S$ Inductive case (which itself has two subcases)

$T \xrightarrow{\text{T.delete(c)}} S$ Inductive case

Goal: prove no case-insentitive duplicates

Property $P(X) = \neg\exists c_1, c_2 \in X$ [toLowerCase(c_1) = toLowerCase(c_2)]

Prove P(S); that is: $\neg\exists c_1, c_2 \in S$ [toLowerCase(c_1) = toLowerCase(c_2)]

How might S have been made?

Consider two possibilities for how S was made: by the constructor, or by `insert`

Base case: S = { }, (S was made by the constructor):

property holds (vacuously true)

Inductive case (S was made by a call of the form "T.insert(c)"):

Show: P(S), that is, $\neg\exists c_1, c_2 \in X$ [toLowerCase(c_1) = toLowerCase(c_2)]

where S = T.insert(c)

= "if ($\exists c_5 \in T$ s.t. toLowerCase(c_5) = toLowerCase(c))
then T **else** T U {c}"

The value for S came from the specification of insert, applied to T.insert(c):

// modifies: this

// effects: $this_{post}$ = ***if (***$\exists c_1 \in$***S s.t. toLowerCase(***c_1***) = toLowerCase(c))***

then $this_{pre}$

else $this_{pre}$ ***U {c}***

public void insert (char c);

(Inductive case is continued on the next slide.)

Goal: no case-insensitive duplicates.

Inductive case: S = T.insert(c)

Goal (from previous slide):

Assume: P(T), that is: $\neg\exists c_3, c_4 \in T$ [toLowerCase(c_3) = toLowerCase(c_4)]

Show: P(S), that is: $\neg\exists c_1, c_2 \in S$ [toLowerCase(c_1) = toLowerCase(c_2)]

where S = T.insert(c)

= "if ($\exists c_5 \in T$ s.t. toLowerCase(c_5) = toLowerCase(c))

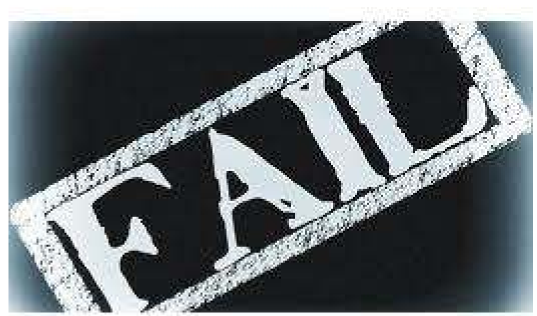
then T **else** T U {c}"

Consider the two possibilities for S (from "if ... then T else T U {c}"):

1. If S = T, then we have not introduced a duplicate (duh)
   and T had no duplicate to begin with

2. If S = T U {c}, then P(S) holds because of the if statement in the specification and the definition of union

Therefore, P(S) holds



Attempt #2

Goal: prove that no LetterSet contains upper- and lower-case versions of a letter

Property P(X) = $\neg\exists c_1, c_2 \in X$ [toLowerCase(c_1) = toLowerCase(c_2)]

Prove P(S); that is: $\neg\exists c_1, c_2 \in S$ [toLowerCase(c_1) = toLowerCase(c_2)]

Use induction on the size of S.

How big is S?

Size 0 — Base case

Size >0 — Inductive case

Goal: prove no case-insensitive duplicates

Property $P(X) = \neg\exists c_1, c_2 \in X$ [$toLowerCase(c_1) = toLowerCase(c_2)$]

Prove P(S); that is: $\neg\exists c_1, c_2 \in S$ [$toLowerCase(c_1) = toLowerCase(c_2)$]

How might S have been made?

Consider three possibilities for how S was made: by the constructor, or by **`insert`**, or by **`delete`**

Base case: S = { }, (S was made by the constructor):

property holds (vacuously true)

Inductive case (S was made by a call of the form “T.insert(c)”):

Assume: P(T) for all T such that $|T| < |S|$

Show: P(S), that is, $\neg\exists c_1, c_2 \in X$ [$toLowerCase(c_1) = toLowerCase(c_2)$]

Tricky because it’s possible that $|T.insert(c)| = |T|$

Inductive case (S was made by a call of the form “T.delete(c)”):

Assume: P(T) for all T such that $|T| > |S|$

Show: P(S), that is, $\neg\exists c_1, c_2 \in X$ [$toLowerCase(c_1) = toLowerCase(c_2)$]

Proof by induction over the computation

Any LetterSet was constructed by a sequence of calls like this:

LetterSet().insert(c1).insert(c2). … .delete(c3).insert(c4)

Assume property holds for this object

Prove property holds for this object

Also prove the base case

Goal: prove that a large enough LetterSet contains two different letters

Property P(X) = |X| > 1 $\Rightarrow$ ($\exists c_1$, $c_2 \in$ X [toLowerCase(c_1) $\neq$ toLowerCase(c_2)])

Prove P(S); that is: |S| > 1 $\Rightarrow$ ($\exists c_1$, $c_2 \in$ S [toLowerCase(c_1) $\neq$ toLowerCase(c_2)])

How might S have been made?

$\xrightarrow{\text{constructor}}$ S   Base case

T $\xrightarrow{\text{T.insert(c)}}$ S   Inductive case

(ignore delete(c) to keep the proof short)

Goal: prove that a large enough LetterSet contains two different letters

Property $P(X) = |X| > 1 \Rightarrow (\exists c_1, c_2 \in X\ [\text{toLowerCase}(c_1) \neq \text{toLowerCase}(c_2)])$

Prove P(S)

Two possibilities for how S was made: by the constructor, or by **`insert`**

Base case: S = { }, (S was made by the constructor):

property holds (vacuously true)

Inductive case (S was made by a call of the form “T.insert(c)”):

Assume: P(T), that is, $|T| > 1 \Rightarrow (\exists c_3, c_4 \in T\ [\text{toLowerCase}(c_3) \neq \text{toLowerCase}(c_4)])$

Show: P(S), that is, $|S| > 1 \Rightarrow (\exists c_1, c_2 \in S\ [\text{toLowerCase}(c_1) \neq \text{toLowerCase}(c_2)])$

where S = T.insert(c)

= “if $(\exists c_5 \in T$ s.t. $\text{toLowerCase}(c_5) = \text{toLowerCase}(c))$

then T **else** T U {c}”

The value for S came from the specification of insert, applied to T.insert(c):

// modifies: this

// effects: this_{post} = ***if*** $(\exists c_1 \in S$ ***s.t. toLowerCase(***c_1***) = toLowerCase(c))***

then $\textit{this}_{pre}$

else $\textit{this}_{pre}$ ***U {c}***

public void insert (char c);

(Inductive case is continued on the next slide.)

Goal: a large enough LetterSet contains two different letters.

Inductive case: S = T.insert(c)

Goal (from previous slide):

Assume: P(T), that is, $|T| > 1 \Rightarrow (\exists c_3, c_4 \in T$ [toLowerCase(c_3) $\neq$ toLowerCase(c_4)])

Show: P(S), that is, $|S| > 1 \Rightarrow (\exists c_1, c_2 \in S$ [toLowerCase(c_1) $\neq$ toLowerCase(c_2)])

where S = T.insert(c)

= "if ($\exists c_5 \in T$ s.t. toLowerCase(c_5) = toLowerCase(c))

then T **else** T U {c}"

Consider the two possibilities for S (from "if ... then T else T U {c}"):

1. If S = T, then P(S) holds by the induction hypothesis or assumption that P(T)
2. If S = T U {c}, there are three cases to consider:
   - |T| = 0: P(S) holds vacuously, since hypothesis ("|S| > 1") is false
   - |T| ≥ 1: We know that T did not contain a char of toLowerCase(c), so P(S) holds by the meaning of union

   We didn't need to use the induction hypothesis for this case

   - Bonus: |T| > 1: By inductive assumption, T contains different letters, so by the meaning of union, T U {c} also contains different letters

Conclusion

The goal is correct code

A proof is a powerful mechanism for ensuring correctness

Formal reasoning is required if debugging is hard

Inductive proofs are the most effective in computer science

Types of proofs:

- Verify that rep invariant is satisfied (today)
- Verify that the implementation satisfies the spec ("reasoning about code" lectures)
- Verify that client code behaves correctly (today)