CSE 331
Software Design & Implementation

Hal Perkins

Spring 2017

Lecture 2 – Reasoning About Code With Logic

Administrivia

- Discussion board: be sure to post a reply to the welcome message

- Next few lectures: two presentations on the web:
  - Lecture notes
  - Powerpoint slides

  They are complementary and you should understand both of them

- HW1 out later today. Programming logic with no loops. Due Tuesday night, 11 pm.

- Initial office hours schedule posted, and we'll start today. We can adjust if needed – let us know (discussion board would be good)

Notetakers needed

- DRS (disability resources for students) is looking for 2 or more notetakers to help one of your colleagues who is taking the class. If you take good notes and would be willing to help, please see me after class or send a note to the staff mailing list (cse331-staff@cs) so we can get this set up.* Thanks

*And besides being the right thing to do, DRS will provide a letter of recommendation at the end of the quarter to document your service.

Reasoning about code

Determine what facts are true as a program executes

- Under what assumptions

Examples:

- If **`x`** starts positive, then **`y`** is **`0`** when the loop finishes
- Contents of the array that **`arr`** refers to are sorted
- Except at one code point, **`x + y == z`**
- For all instances of **`Node n`**,
  `n.next == null` ∨ **`n.next.prev == n`**
- …

Why do this?

- Essential complement to *testing*, which we will also study
  - Testing: Actual results for some actual inputs
  - Logical reasoning: Reason about whole classes of inputs/states at once ("If `x > 0`, …")
    - *Prove* a program correct (or find bugs trying), or (even better) develop program and proof together to get a program that is correct by construction
    - Understand *why* code is correct

- Stating assumptions is the essence of specification
  - "Callers must not pass `null` as an argument"
  - "Callee will always return an unaliased object"
  - …

Our approach

- Hoare Logic: a 1970s approach to logical reasoning about code
  - For now, consider just variables, assignments, if-statements, while-loops
    - So no objects or methods

- This lecture: The idea, without loops, in 3 passes
  1. High-level intuition of forward and backward reasoning
  2. Precise definition of logical assertions, preconditions, etc.
  3. Definition of weaker/stronger and weakest-precondition

- Next lecture: Loops

Why?

- Programmers rarely "use Hoare logic" in this much detail
  - For simple snippets of code, it's overkill
  - Gets very complicated with objects and aliasing
  - But can be very useful to develop and reason about loops and data with subtle *invariants*
    - Examples: Homework 0, Homework 2

- Also it's an ideal setting for the right logical foundations
  - How can logic "talk about" program states?
  - How does code execution "change what is true"?
  - What do "weaker" and "stronger" mean?

This is all essential for *specifying library-interfaces*, which *does* happen All the Time in The Real World® (coming lectures)

Example

Forward reasoning:

- Suppose we initially know (or assume) `w > 0`

```
// w > 0
x = 17;
// w > 0  ∧  x == 17
y = 42;
// w > 0  ∧  x == 17 ∧  y == 42
z = w + x + y;
// w > 0 ∧ x == 17 ∧ y == 42 ∧ z > 59
…
```

- Then we know various things after, including `z > 59`

Example

Backward reasoning:

- Suppose we want `z` to be negative at the end

```
// w + 17 + 42 < 0
x = 17;
// w + x + 42 < 0
y = 42;
// w + x + y < 0
z = w + x + y;
// z < 0
```

- Then we know initially we need to know/assume `w < -59`
    - Necessary and sufficient

Forward vs. Backward, Part 1

- Forward reasoning:
  - Determine what follows from initial assumptions
  - Most useful for *maintaining an invariant*

- Backward reasoning
  - Determine sufficient conditions for a certain result
    - If result desired, the assumptions suffice for correctness
    - If result undesired, the assumptions suffice to trigger bug

Forward vs. Backward, Part 2

- Forward reasoning:
  - Simulates the code (for many “inputs” “at once”)
  - Often more intuitive
  - But introduces [many] facts irrelevant to a goal

- Backward reasoning
  - Often more useful: Understand what each part of the code contributes toward the goal
  - “Thinking backwards” takes practice but gives you a powerful new way to reason about programs

Conditionals

```
// initial assumptions
if(…) {
  … // also know test evaluated to true
} else {
  … // also know test evaluated to false
}
// either branch could have executed
```

Two key ideas:

1. The precondition for each branch includes information about the result of the test-expression
2. The overall postcondition is the disjunction ("or") of the postcondition of the branches

Example (Forward)

Assume initially **`x >= 0`**

```
// x >= 0
z = 0;
// x >= 0 ∧ z == 0
if(x != 0) {
  // x >= 0 ∧ z == 0 ∧ x != 0 (so x > 0)
  z = x;
  // … ∧ z > 0
} else {
  // x >= 0 ∧ z == 0 ∧ !(x!=0) (so x == 0)
  z = x + 1;
  // … ∧ z == 1
}
// ( … ∧ z > 0) v (… ∧ z == 1)  (so z > 0)
```

Our approach

- Hoare Logic, a 1970s approach to logical reasoning about code
  - [Named after its inventor, Tony Hoare]
  - Considering just variables, assignments, if-statements, while-loops
    - So no objects or methods

- This lecture: The idea, without loops, in 3 passes
  1. High-level intuition of forward and backward reasoning
  2. Precise definition of logical assertions, preconditions, etc.
  3. Definition of weaker/stronger and weakest-precondition

- Next lecture: Loops

Some notation and terminology

- The “assumption” before some code is the precondition
- The “what holds after (given assumption)” is the postcondition

- Instead of writing pre/postconditions after //, write them in {…}
  - This is not Java
  - How Hoare logic has been written “on paper” for 40ish years

```
{ w < -59 }
x = 17;
{ w + x < -42 }
```

  - In pre/postconditions, = is equality, not assignment
    - Math’s “=”, which for numbers is Java’s ==

```
{ w > 0  ∧  x = 17 }
y = 42;
{ w > 0  ∧  x = 17 ∧  y = 42 }
```

What an assertion means

- An *assertion* (pre/postcondition) is a logical formula that can refer to program state (e.g., contents of variables)
- A *program state* is something that “given” a variable can “tell you” its contents
  - Or any expression that has no *side-effects*
- An assertion *holds* for a program state, if evaluating using the program state produces *true*
  - Evaluating a program variable produces its contents in the state
  - Can think of an assertion as representing the *set* of (exactly the) states for which it holds

A Hoare Triple

- A Hoare triple is two assertions and one piece of code:

  `{P} S {Q}`

  – *P* the precondition
  – *S* the code (statement)
  – *Q* the postcondition

- A Hoare triple `{P} S {Q}` is (by definition) valid if:
  – For all states for which *P* holds, executing *S* always produces a state for which *Q* holds
  – Less formally: If *P* is true before *S*, then *Q* must be true after
  – Else the Hoare triple is invalid

Examples

Valid or invalid?

- (Assume all variables are integers without overflow)

- `{x != 0} y = x*x; {y > 0}`
- `{z != 1} y = z*z; {y != z}`
- `{x >= 0} y = 2*x; {y > x}`
- `{true} (if(x > 7) {y=4;} else {y=3;}) {y < 5}`
- `{true} (x = y; z = x;) {y=z}`
- `{x=7 ∧ y=5}`
  `(tmp=x; x=tmp; y=x;)`
  `{y=7 ∧ x=5}`

Examples

Valid or invalid?

– (Assume all variables are integers without overflow)

- `{x != 0} y = x*x; {y > 0}` valid
- `{z != 1} y = z*z; {y != z}` invalid
- `{x >= 0} y = 2*x; {y > x}` invalid
- `{true} (if(x > 7) {y=4;} else {y=3;}) {y < 5}` valid
- `{true} (x = y; z = x;) {y=z}` valid
- `{x=7 ∧ y=5}` invalid

  `(tmp=x; x=tmp; y=x;)`

  `{y=7 ∧ x=5}`

Aside: assert in Java

- An assertion in Java is a statement with a Java expression, e.g.,

```
assert x > 0 && y < x;
```

- Similar to our assertions
  - Evaluate using a program state to get **`true`** or **`false`**
  - Uses Java syntax

- In Java, this is a run-time thing: Run the code and raise an exception if assertion is violated
  - Unless assertion-checking is disabled
  - Later course topic

- This week: we are reasoning about the code, not running it on some input

The general rules

- So far: Decided if a Hoare triple was valid by using our understanding of programming constructs

- Now: For each kind of construct there is a general rule
  - A rule for assignment statements
  - A rule for two statements in sequence
  - A rule for conditionals
  - [next lecture:] A rule for loops
  - …

Basic rule: Assignment

```
{P} x = e; {Q}
```

- Let **`Q′`** be like **`Q`** except replace every **`x`** with **`e`**
- Triple is valid if:

  For all program states, if **`P`** holds, then **`Q′`** holds
  - That is, **`P`** implies **`Q′`**, written **`P => Q′`**

- Example: **`{z > 34} y=z+1; {y > 1}`**
  - **`Q′ is {z+1 > 1}`**

Combining rule: Sequence

`{P} S1;S2 {Q}`

- Triple is valid if and only if there is an assertion `R` such that
  - `{P}S1{R}` is valid, and
  - `{R}S2{Q}` is valid

- Example: `{z >= 1} y=z+1; w=y*y; {w > y}` (integers)
  - Let `R` be `{y > 1}`
  - Show `{z >= 1} y=z+1; {y > 1}`
    - Use rule for assignments: `z >= 1` implies `z+1 > 1`
  - Show `{y > 1} w=y*y; {w > y}`
    - Use rule for assignments: `y > 1` implies `y*y > y`

Combining rule: Conditional

`{P} if(b) S1 else S2 {Q}`

- Triple is valid if and only if there are assertions `Q1,Q2` such that
  - `{P ∧ b}S1{Q1}` is valid, and
  - `{P ∧ !b}S2{Q2}` is valid, and
  - `Q1 ∨ Q2` implies `Q`

- Example: `{true} (if(x > 7) y=x; else y=20;) {y > 5}`
  - Let `Q1` be `{y > 7}` (other choices work too)
  - Let `Q2` be `{y = 20}` (other choices work too)
  - Use assignment rule to show `{true ∧ x > 7}y=x;{y>7}`
  - Use assignment rule to show `{true ∧ x <= 7}y=20;{y=20}`
  - Indicate `y>7 ∨ y=20` implies `y>5`

Our approach

- Hoare Logic, a 1970s approach to logical reasoning about code
  - Considering just variables, assignments, if-statements, while-loops
    - So no objects or methods

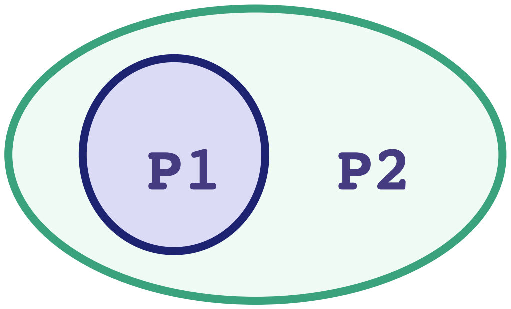
- This lecture: The idea, without loops, in 3 passes
  1. High-level intuition of forward and backward reasoning
  2. Precise definition of logical assertions, preconditions, etc.
  3. Definition of weaker/stronger and weakest-precondition

- Next lecture: Loops

Weaker vs. Stronger

If P1 implies P2 (written P1 => P2), then:

- P1 is stronger than P2
- P2 is weaker than P1



- Whenever P1 holds, P2 also holds
- So it is more (or at least as) "difficult" to satisfy P1
  - The program states where P1 holds are a subset of the program states where P2 holds
- So P1 puts more constraints on program states
- So it's a stronger set of obligations/requirements

Examples

- `x = 17` is stronger than `x > 0`
- `x is prime` is neither stronger nor weaker than `x is odd`
- `x is prime and x > 2` is stronger than
  `x is odd and x > 2`
- …

Why this matters to us

- Suppose:
  - `{P}S{Q}`, and
  - `P` is weaker than some `P1`, and
  - `Q` is stronger than some `Q1`
- Then: `{P1}S{Q}` and `{P}S{Q1}` and `{P1}S{Q1}`
- Example:
  - `P` is `x >= 0`
  - `P1` is `x > 0`
  - `S` is `y = x+1`
  - `Q` is `y > 0`
  - `Q1` is `y >= 0`

So…

- For backward reasoning, if we want `{P}S{Q}`, we could instead:
  - Show `{P1}S{Q}`, and
  - Show `P => P1`

- Better, we could just show `{P2}S{Q}` where `P2` is the weakest precondition of `Q` for `S`
  - Weakest means the most lenient assumptions such that `Q` will hold after executing `S`
  - Any precondition `P` such that `{P}S{Q}` is valid will be stronger than `P2`, i.e., `P => P2`

- Amazing (?): Without loops/methods, for any `S` and `Q`, there exists a unique weakest precondition, written wp(`S`,`Q`)
  - Like our general rules with backward reasoning

Weakest preconditions

- wp(**`x = e;`**, **`Q`**) is **`Q`** with each **`x`** replaced by **`e`**
  - Example: wp(**`x = y*y;`**, **`x > 4`**) = **`y*y > 4`**, i.e., **`|y| > 2`**

- wp(**`S1;S2`**, **`Q`**) is wp(**`S1`**,wp(**`S2`**,**`Q`**))
  - i.e., let **`R`** be wp(**`S2`**,**`Q`**) and overall wp is wp(**`S1`**,**`R`**)
  - Example: wp(**`(y=x+1; z=y+1;)`**, **`z > 2`**) =
    `(x + 1)+1 > 2`, i.e., **`x > 0`**

- wp(**`if b S1 else S2`**, **`Q`**) is this logic formula:

  `(b` $\land$ wp(**`S1`**,**`Q`**)) **`v`** **`(!b`** $\land$ wp(**`S2`**,**`Q`**))

  - (In any state, b will evaluate to either true or false…)
  - (You can sometimes then simplify the result)

Simple examples

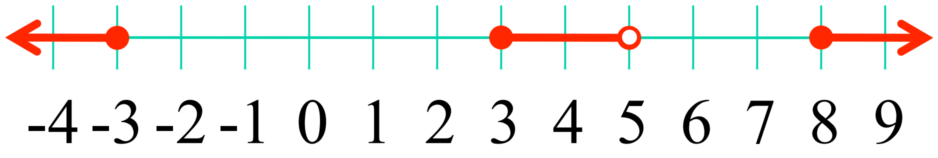
- If `S` is `x = y*y` and `Q` is `x > 4`,
  then wp(`S`,`Q`) is `y*y > 4`, i.e., `|y| > 2`

- If `S` is `y = x + 1; z = y – 3;` and `Q` is `z = 10`,
  then wp(`S`,`Q`) …
  = wp(`y = x + 1; z = y – 3;`, `z = 10`)
  = wp(`y = x + 1;`, wp(`z = y – 3;`, `z = 10`))
  = wp(`y = x + 1;`, `y-3 = 10`)
  = wp(`y = x + 1;`, `y = 13`)
  = `x+1 = 13`
  = `x = 12`

Bigger example

```
S is if (x < 5) {
         x = x*x;
       } else {
         x = x+1;
       }
Q is x >= 9
```

wp(S, `x >= 9`)
 = (`x < 5` ∧ wp(`x = x*x;`, `x >= 9`))
 ∨ (`x >= 5` ∧ wp(`x = x+1;`, `x >= 9`))
 = (`x < 5` ∧ `x*x >= 9`)
 ∨ (`x >= 5` ∧ `x+1 >= 9`)
 = (`x <= -3`) ∨ (`x >= 3` ∧ `x < 5`)
 ∨ (`x >= 8`)



If-statements review

Forward reasoning

```
{P}
if B
   {P ∧ B}
   S1
   {Q1}
else
   {P ∧ !B}
   S2
   {Q2}
{Q1 ∨ Q2}
```

Backward reasoning

```
{ (B ∧ wp(S1, Q))
   ∨  (!B ∧ wp(S2, Q)) }
if B
   {wp(S1, Q)}
   S1
   {Q}
else
   {wp(S2, Q)}
   S2
   {Q}
{Q}
```

“Correct”

- If wp(`S`,`Q`) is **`true`**, then executing `S` will always produce a state where `Q` holds
  - **`true`** holds for every program state

One more issue

- With forward reasoning, there is a problem with assignment:
  - Changing a variable can affect other assumptions

- Example:

```
{true}
w=x+y;
{w = x + y;}
x=4;
{w = x + y ∧ x = 4}
y=3;
{w = x + y ∧ x = 4 ∧ y = 3}
```

But clearly we do not know **`w=7`**!

The fix

- When you assign to a variable, you need to replace all other uses of the variable in the post-condition with a different variable
  - So you refer to the “old contents”

- Corrected example:

```
{true}
w=x+y;
{w = x + y;}
x=4;
{w = x1 + y ∧ x = 4}
y=3;
{w = x1 + y1 ∧ x = 4 ∧ y = 3}
```

Useful example: swap

- Swap contents
  - Give a name to initial contents so we can refer to them in the post-condition
  - Just in the formulas: these "names" are not in the program
  - Use these extra variables to avoid "forgetting" "connections"

```
{x = x_pre ∧ y = y_pre}
tmp = x;
{x = x_pre ∧ y = y_pre ∧ tmp=x}
x = y;
{x = y ∧ y = y_pre ∧ tmp=x_pre}
y = tmp;
{x = y_pre ∧ y = tmp ∧ tmp=x_pre}
```