CSE332: Data Abstractions

Lecture 9: B Trees

Dan Grossman

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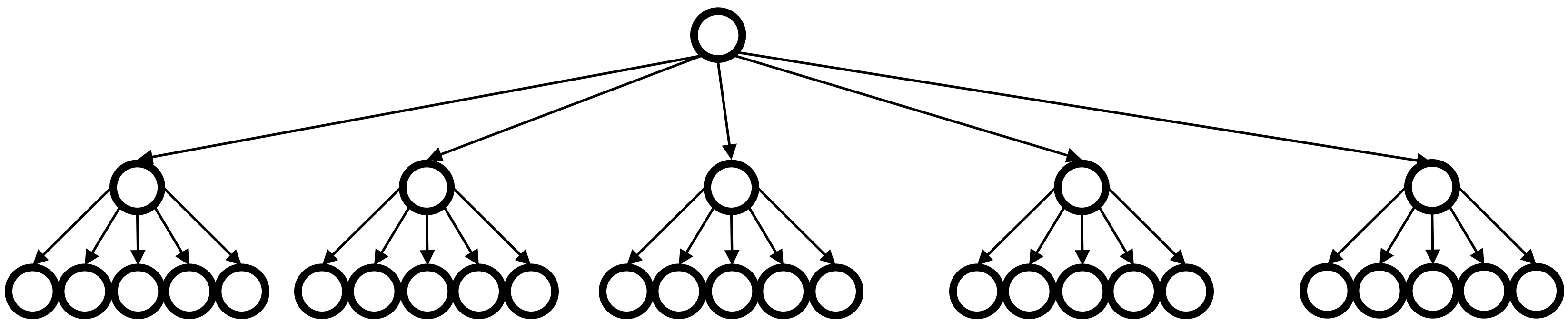
- Problem: A dictionary with so much data most of it is on disk

- Desire: A balanced tree (logarithmic height) that is even shallower than AVL trees so that we can minimize disk accesses and exploit disk-block size

- Key idea: Increase the branching factor of our tree

M-ary Search Tree

- Build some sort of search tree with branching factor *M*:
  - Have an array of sorted children (**`Node[]`**)
  - Choose *M* to fit snugly into a disk block (1 access for array)



Perfect tree of height *h* has $(M^{h+1}-1)/(M-1)$ nodes (textbook, page 4)

\# hops for **`find`**: If balanced, then $\log_M n$ instead of $\log_2 n$

- If M=256, that's an 8x improvement
- Example: $M = 256$ and $n = 2^{40}$ that's 5 instead of 40

Runtime of **`find`** if balanced: $O(\log_M n \log_2 M)$ (binary search children)

Problems with M-ary search trees

- What should the order property be?
- How would you rebalance (ideally without more disk accesses)?
- Any "useful" data at the internal nodes takes up disk-block space without being used by finds moving past it

So let's use the branching-factor idea, but for a *different kind of balanced tree*

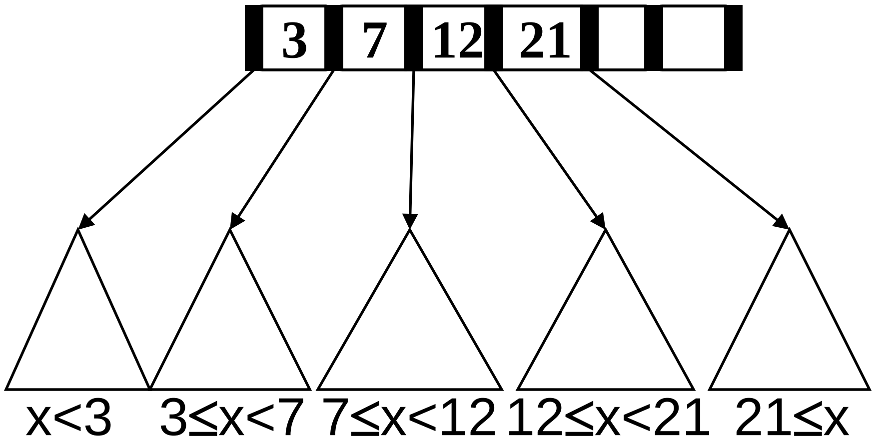
- Not a binary search tree
- But still logarithmic height for any $M > 2$

B+ Trees (we and the book say "B Trees")

- Each internal node has room for up to *M-1* keys and *M* children
  - No other data: all data at leaves!
- Order property:

  Subtree **between** keys *x* and *y*

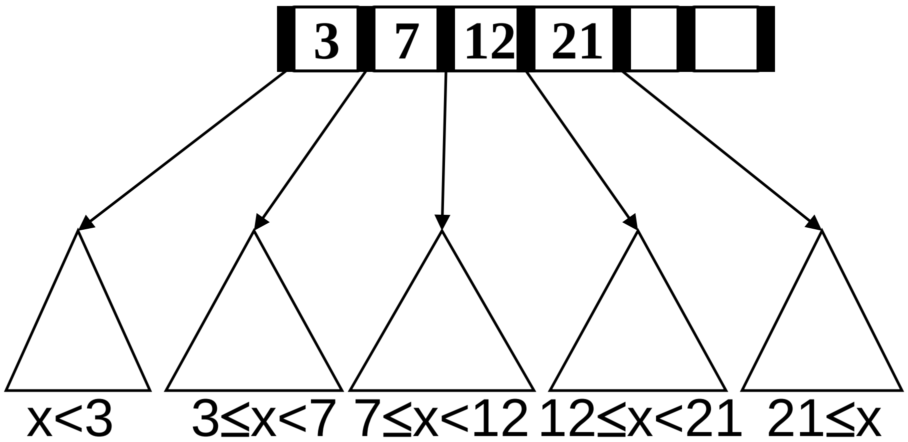
  contains data with keys ≥ ***x*** and < ***y***
- Leaf nodes have up to *L* sorted data items



As usual, we wont show the "along for the ride" data in our examples

- Remember no data at non-leaves

Find



- This is a new kind of tree
  - We are used to data at internal nodes

- **`find`** is still an easy root-to-leaf recursive algorithm
  - At each internal node, do binary search on the ≤ M-1 keys
  - At the leaf, do binary search on the ≤ L data items

- To get logarithmic running time, we need a balance condition…

Structure Properties

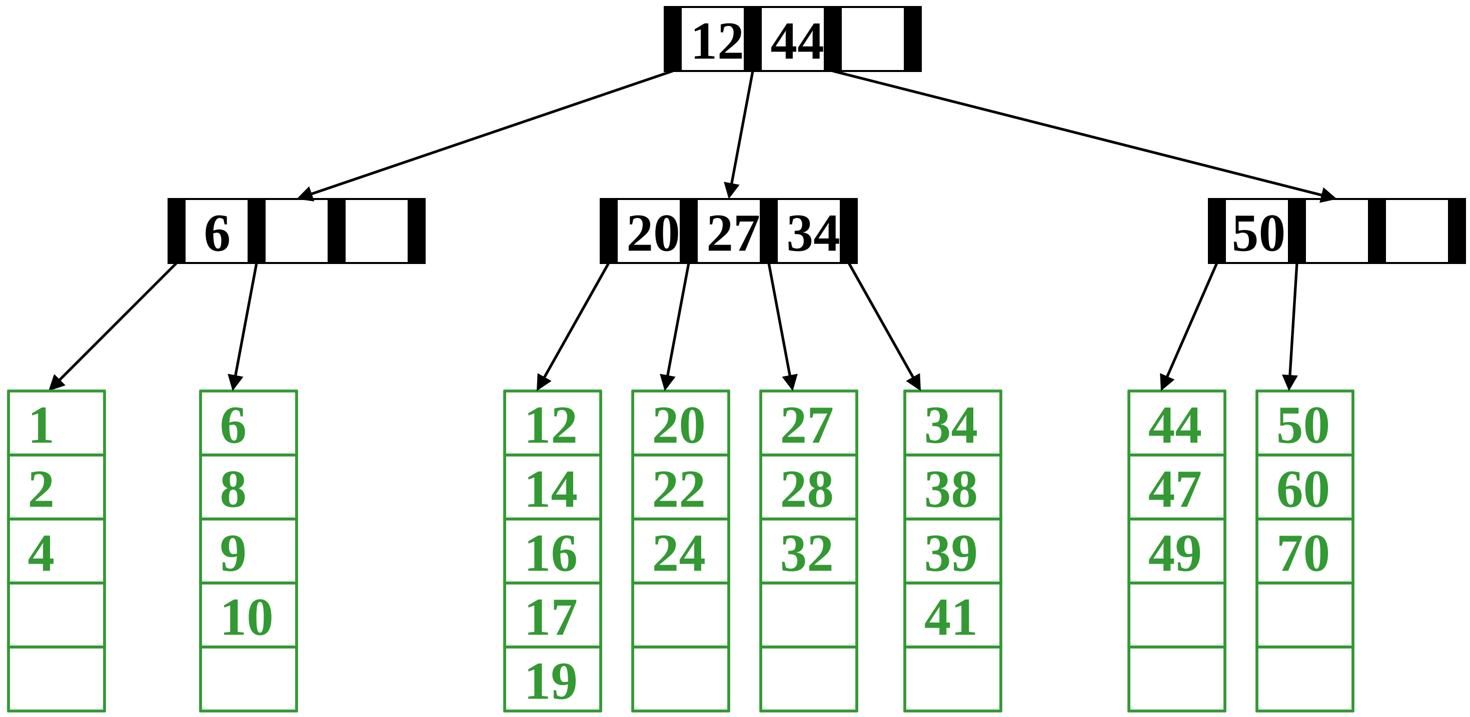
- Root (special case)
  - If tree has $\leq L$ items, root is a leaf (very strange case)
  - Else has between 2 and M children
- Internal nodes
  - Have between $\lceil M/2 \rceil$ and M children, i.e., *at least half full*
- Leaf nodes
  - *All leaves at the same depth*
  - Have between $\lceil L/2 \rceil$ and L data items, i.e., *at least half full*

(Any $M > 2$ and L will work; picked based on disk-block size)

Example

Suppose *M*=4 (max # children) and *L*=5 (max # at leaf)

- All internal nodes have at least 2 children
- All leaves have at least 3 data items (only showing keys)
- All leaves at same depth



Balanced enough

Not hard to show height *h* is logarithmic in number of data items *n*

- Recall $M > 2$ (if $M = 2$, then a list tree is legal – no good!)
- Because all nodes are at least half full (except root may have only 2 children) and all leaves are at the same level, the minimum number of data items *n* for a height $h>0$ tree is…

$$n \geq \underbrace{2 \lceil M/2 \rceil^{h-1}}_{\text{minimum number of leaves}} \underbrace{\lceil L/2 \rceil}_{\text{minimum data per leaf}}$$

Exponential in height because $\lceil M/2 \rceil > 1$

B-Tree vs. AVL Tree

Suppose we have 100,000,000 items

- Maximum height of AVL tree?

- Maximum height of B tree with M=128 and L=64?

B-Tree vs. AVL Tree

Suppose we have 100,000,000 items

- Maximum height of AVL tree?
  - Recall S(h) = 1 + S(h-1) + S(h-2)
  - lecture7.xlsx reports: 37

- Maximum height of B tree with M=128 and L=64?
  - Recall $(2 \lceil M/2 \rceil^{h-1}) \lceil L/2 \rceil$
  - lecture9.xlsx reports: 5 (and 4 is more likely)
  - Also not difficult to compute via algebra

Disk Friendliness

Why are B trees so disk friendly?

- Many keys stored in one internal node
  - All brought into memory in one disk access
  - Pick M wisely. Example: block=1KB, then M=128
  - Makes the binary search over M-1 keys totally worth it (insignificant compared to disk access times)

- Internal nodes contain only keys
  - Any **`find`** wants only one data item
  - So bring only one leaf of data items into memory
  - Data-item size does not affect value for M

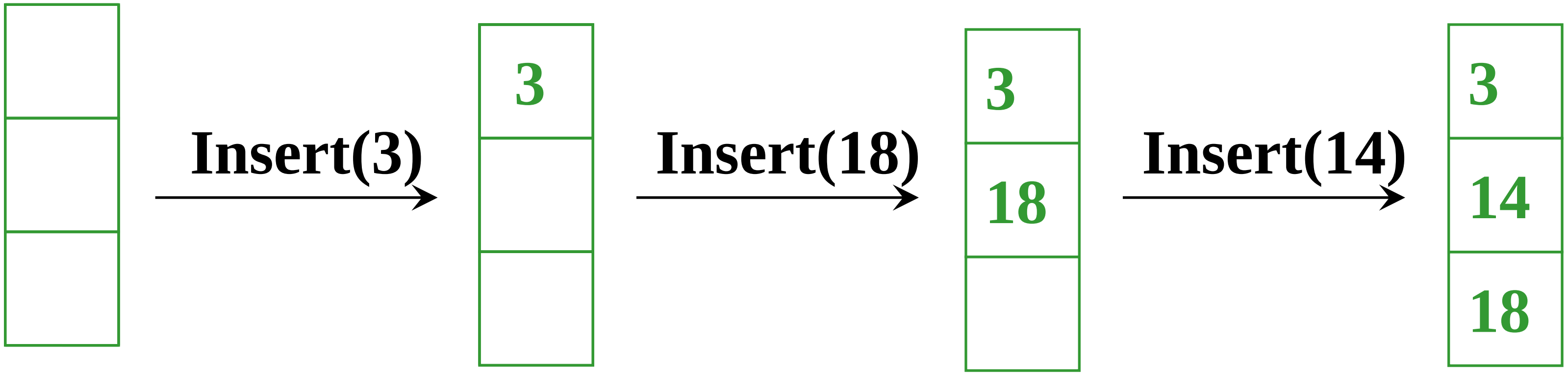
Maintaining balance

- So this seems like a great data structure (and it is)

- But still need to implement the other dictionary operations
  - **`insert`**
  - **`delete`**

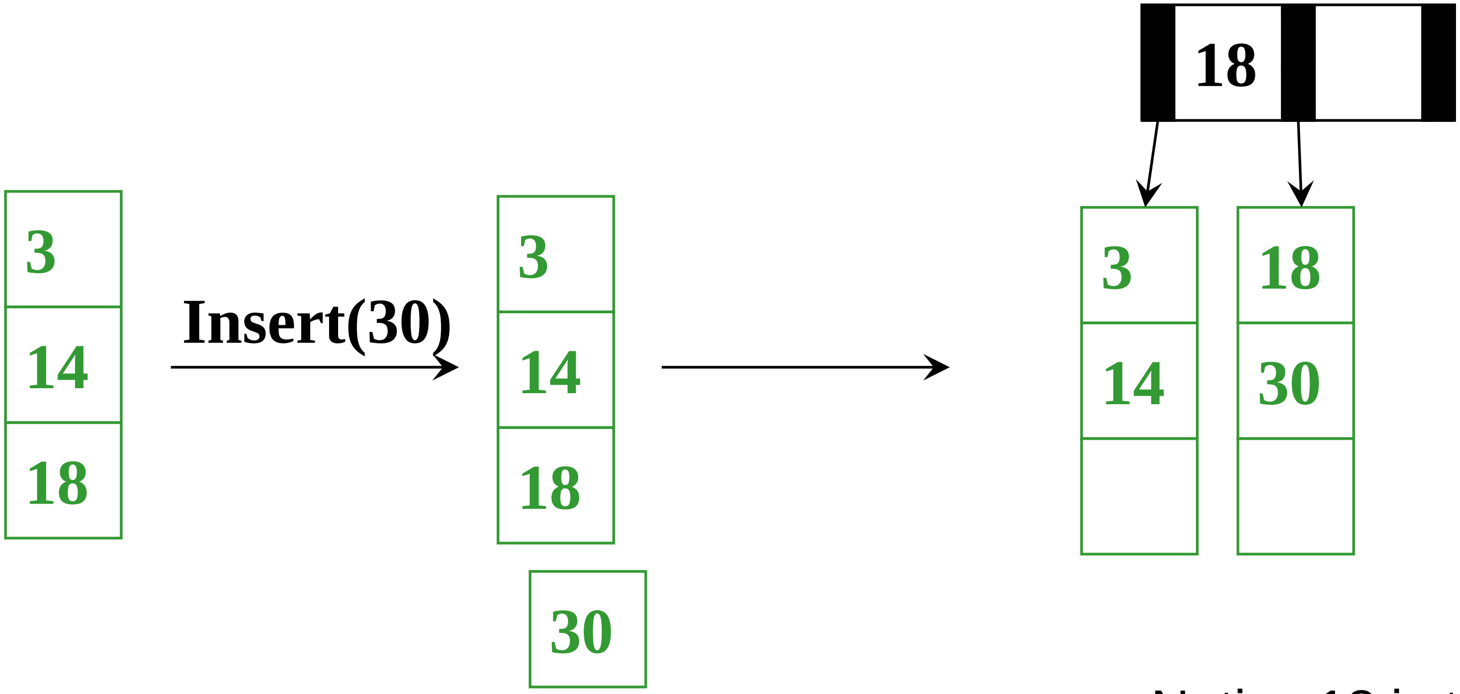
- As with AVL trees, hard part is maintaining structure properties
  - Example: for **`insert`**, there might not be room at the correct leaf

Building a B-Tree (insertions)



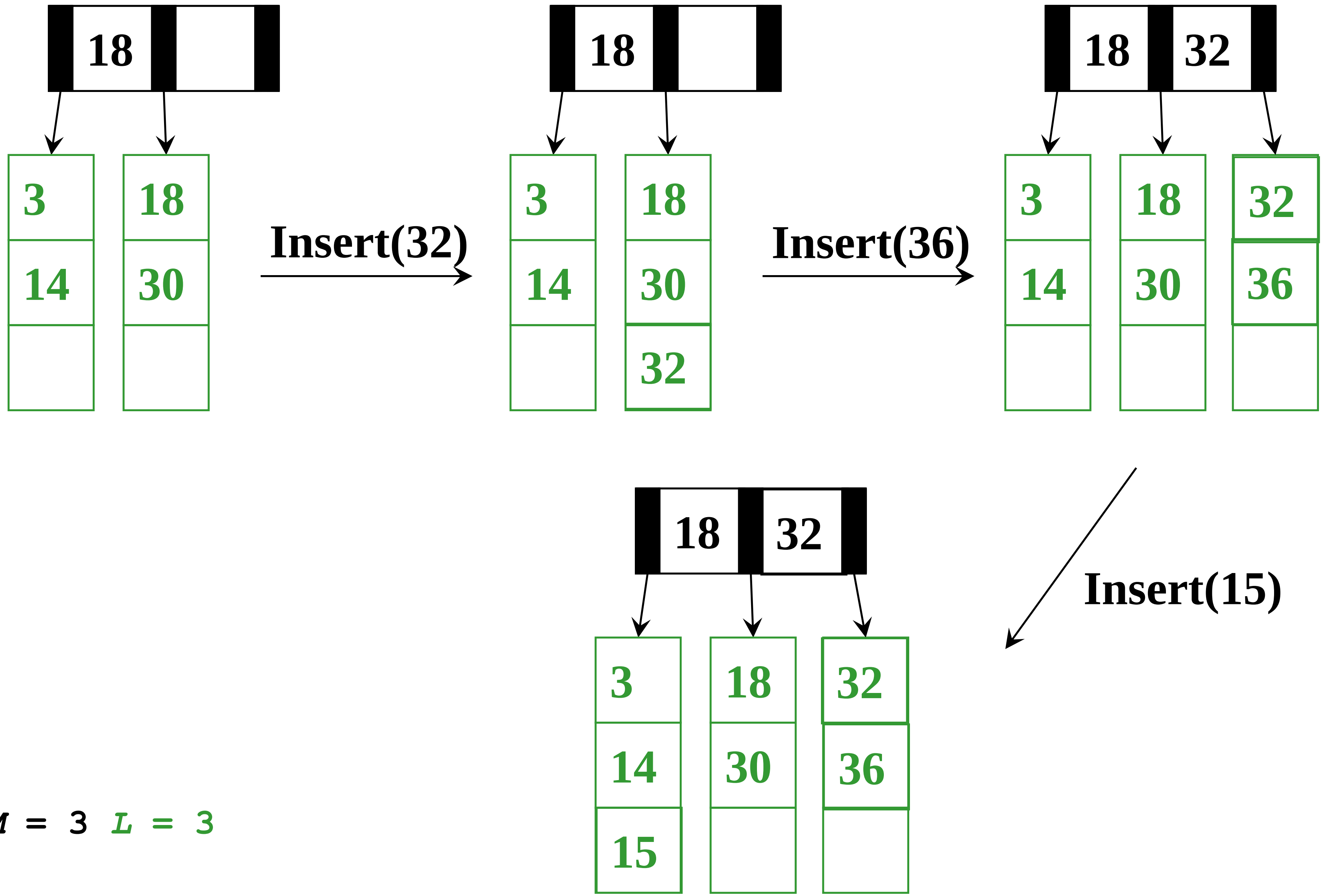
The empty B-Tree (1 empty leaf)

M **= 3** ***L*** **= 3**

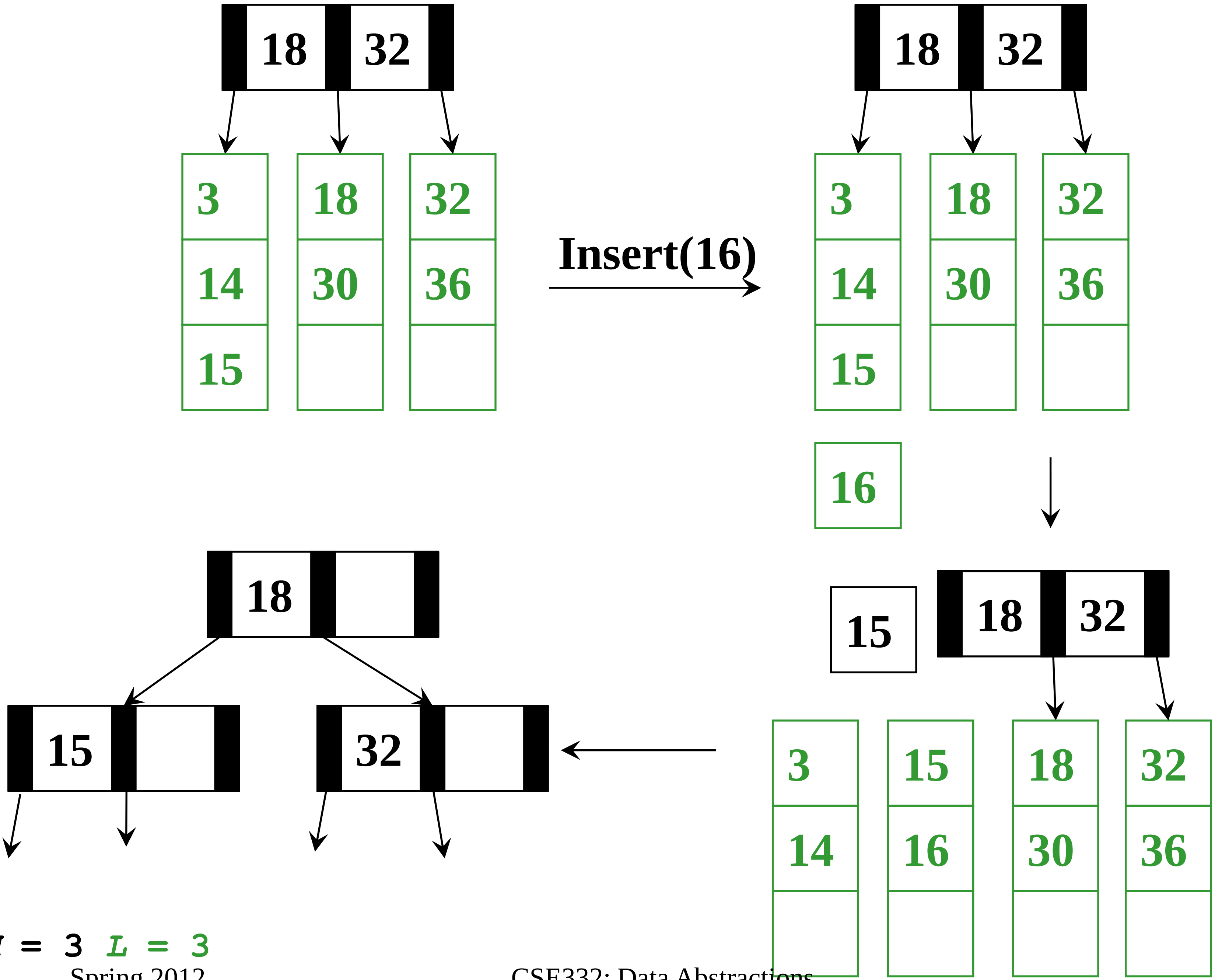


Notice 18 is the smallest key in the right child

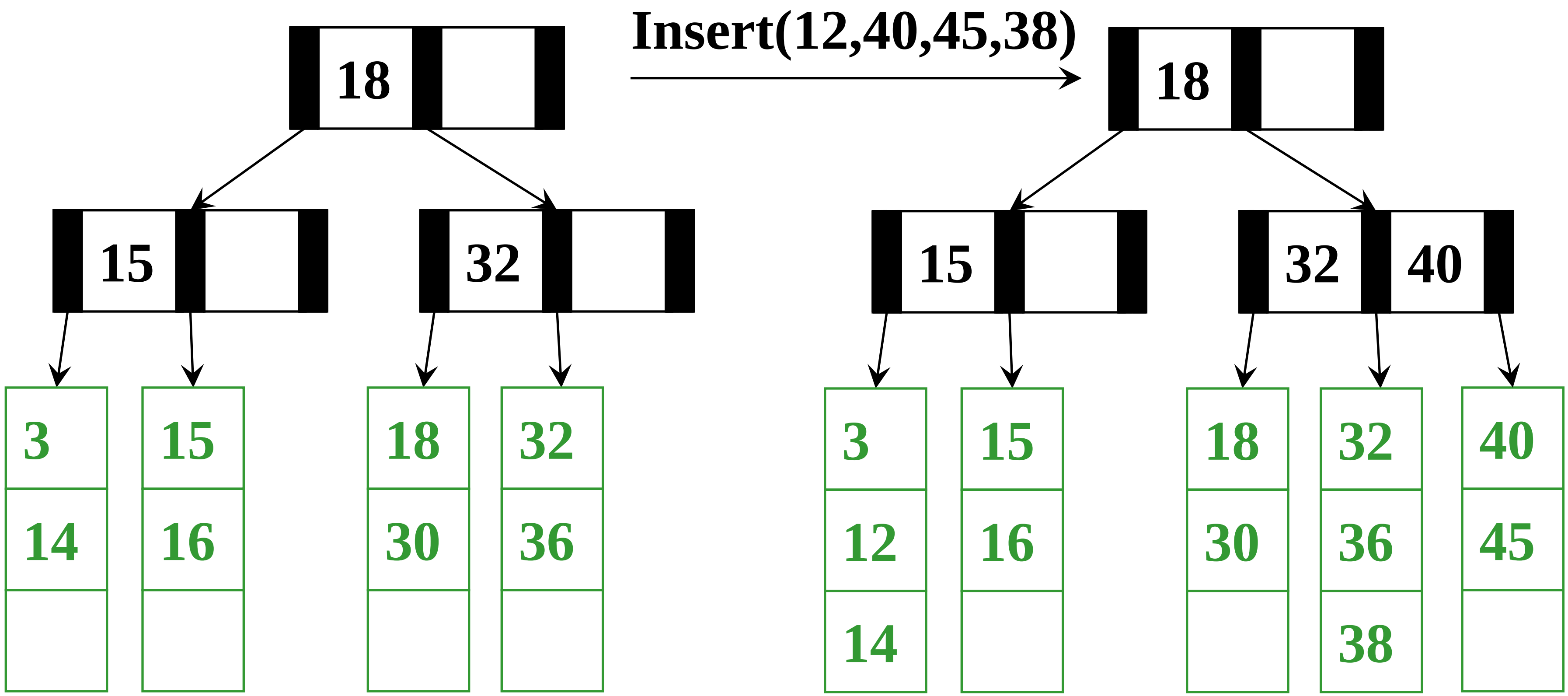
M = 3 **L = 3**


18
3
14
18
30
Insert(32)
18
3
14
18
30
32
Insert(36)
18
32
3
14
18
30
32
36
Insert(15)
18
32
3
14
15
18
30
32
36

`M = 3` `L = 3`


18
32
3
14
15
18
30
32
36
Insert(16)
3
14
15
18
30
32
36
16
15
18
32
3
14
15
16
18
30
32
36
18
15
32

M = 3 **L = 3**



Insert(12,40,45,38)
18
15
32
3
14
15
16
18
30
32
36
18
15
32
40
3
12
14
15
16
18
30
32
36
38
40
45

M = 3 *L = 3*

Insertion Algorithm, part 1

1. Insert the data in its leaf in sorted order

2. If the leaf now has L+1 items, *overflow!*
   - Split the leaf into two nodes:
     - Original leaf with $\lceil (L+1)/2 \rceil$ smaller items
     - New leaf with $\lfloor (L+1)/2 \rfloor = \lceil L/2 \rceil$ larger items
   - Attach the new child to the parent
     - Adding new key to parent in sorted order

3. If step (2) caused the parent to have M+1 children, *overflow!*
   - *…*

Insertion Algorithm, continued

3. If an internal node has M+1 children
   - Split the node into two nodes
     - Original node with $\lceil (M+1)/2 \rceil$ smaller items
     - New node with $\lfloor (M+1)/2 \rfloor = \lceil M/2 \rceil$ larger items
   - Attach the new child to the parent
     - Adding new key to parent in sorted order

Splitting at a node (step 3) could make the parent overflow too

- So repeat step 3 up the tree until a node does not overflow
- If the root overflows, make a new root with two children
  - This is the only case that increases the tree height

Worst-Case Efficiency of Insert

- Find correct leaf: $O(\mathbf{log_2}\ M\ \mathbf{log}_M\ n)$
- Insert in leaf: $O(L)$
- Split leaf: $O(L)$
- Split parents up to root: $O(M\ \mathbf{log}_M\ n)$

Total: $O(L + M\ \mathbf{log}_M\ n)$

But it's not that bad:

- Splits are uncommon (only required when a node is FULL, M and L can be fairly large, and new leaves/nodes after split are half-empty)
- Disk accesses are the name of the game: $O(\mathbf{log}_M\ n)$