CSE 332 Data Abstractions: Graphs and Graph Traversals

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Summer 2012

Some course stuff and a humorous story

GRADES, MIDTERMS, AND IT SEEMED LIKE A GOOD IDEA

The Midterm

It was too long—I admit that

If it helps, this was the first exam I have ever written

Even still, I apologize

You All Did Great

I am more than pleased with your performances on the midterm

The points you missed were clearly due to time constraints and stresses

You showed me you know the material

Good job!

How Grades are Calculated

Many (if not most) CSE major courses use curving to determine final grades

- Homework and exam grades are used as indicators and are adjusted as necessary
- Example:
  A student who does excellent on homework and projects (and goes beyond) will get a grade bumped up even if his/her exam scores are poorer

My Experiences as a Teacher

Timed exams are problematic

- Some of the best students I have known did not do great on exams

The more examples of student work that one sees, the more learning becomes evident

- Even partial effort/incomplete work tells a lot
- Unfortunately, this means losing points

The above leads to missing points

- All students (even myself back in the day) care about points

My Repeated Mistake

As a teacher, I should talk more about how points get transformed into a final grade

I learned this lesson my first year as a TA…

… and indirectly caused the undergraduate CSE servers to crash

It Seemed Like a Good Idea at the Time

At the annual CS education conference (SIGCSE), there is a special panel about teaching mistakes and learning from them

I contributed a story at SIGCSE 2009:

http://faculty.washington.edu/deibel/presentations/sigcse09-good-idea/deibel-seemed-good-idea-sigcse-2009.ppt

My Promises

I know you will miss points

If you do the work in the class and put in the effort, you will earn more than a passing grade

As long as you show evidence of learning, you will earn a good grade regardless

What This Means For You

Keep up the good work

Do not obsess over points

The final will be less intense

That was fun but you are here for learning…

BACK TO CSE 332 AND GRAPH [THEORY]

Where We Are

We have learned about the essential ADTs and data structures:

- Regular and Circular Arrays (dynamic sizing)
- Linked Lists
- Stacks, Queues, Priority Queues
- Heaps
- Unbalanced and Balanced Search Trees

We have also learned important algorithms

- Tree traversals
- Floyd's Method
- Sorting algorithms

Where We Are Going

Less generalized data structures and ADTs

More on algorithms and related problems that require constructing data structures to make the solutions efficient

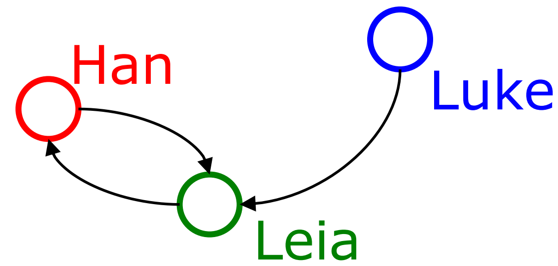
Topics will include:

- Graphs
- Parallelism

Graphs

A graph is a formalism for representing relationships among items

- Very general definition
- Very general concept



A graph is a pair: G = (V, E)

- A set of vertices, also known as nodes: $V = \{v_1, v_2, \ldots, v_n\}$
- A set of edges $E = \{e_1, e_2, \ldots, e_m\}$
  - Each edge e_i is a pair of vertices (v_j, v_k)
  - An edge "connects" the vertices

V = {Han,Leia,Luke}
E = {(Luke,Leia),
(Han,Leia),
(Leia,Han)}

Graphs can be *directed* or *undirected*

A Graph ADT?

We can think of graphs as an ADT

- Operations would inlude **isEdge(v_j,v_k)**
- But it is unclear what the "standard operations" would be for such an ADT

Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms

Many important problems can be solved by:

1. Formulating them in terms of graphs
2. Applying a standard graph algorithm

Some Graphs

For each example, what are the vertices and what are the edges?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
- Course pre-requisites

Core algorithms that work across such domains is why we are CSE

Scratching the Surface

Graphs are a powerful representation and have been studied deeply

Graph theory is a major branch of research in combinatorics and discrete mathematics

Every branch of computer science involves graph theory to some extent

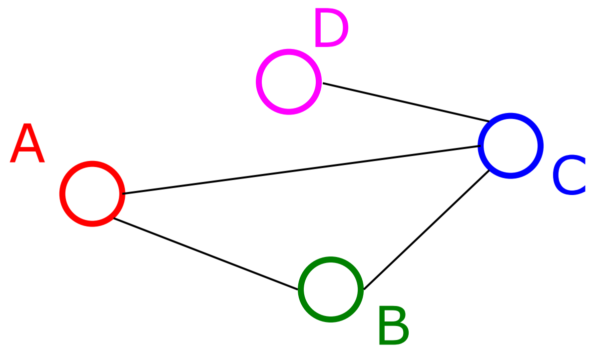
To make formulating graphs easy and standard, we have a lot of *standard terminology* for graphs

GRAPH TERMINOLOGY

Undirected Graphs

In undirected graphs, edges have no specific direction

- Edges are always "two-way"



Thus, $(u, v) \in E$ implies $(v, u) \in E$.

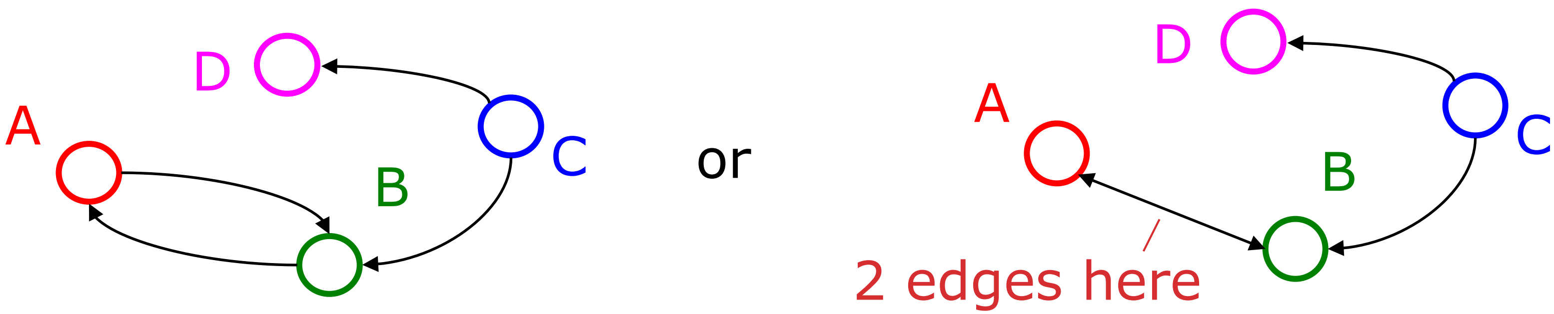
- Only one of these edges needs to be in the set
- The other is implicit, so normalize how you check for it

Degree of a vertex: number of edges containing that vertex

- Put another way: the number of adjacent vertices

Directed Graphs

In directed graphs (or digraphs), edges have direction



Thus, $(u, v) \in E$ does not imply $(v, u) \in E$.

Let $(u, v) \in E$ mean $u \to v$

- Call u the source and v the destination
- In-Degree of a vertex: number of in-bound edges (edges where the vertex is the destination)
- Out-Degree of a vertex: number of out-bound edges (edges where the vertex is the source)

Self-Edges, Connectedness

A self-edge a.k.a. a loop edge is of the form (u, u)

- The use/algorithm usually dictates if a graph has:
  - No self edges
  - Some self edges
  - All self edges

A node can have a(n) degree / in-degree / out-degree of zero

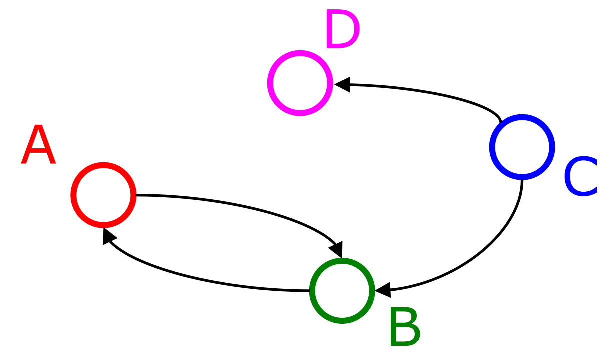
A graph does not have to be connected

- Even if every node has non-zero degree
- More discussion of this to come

More Notation

For a graph $G = (V, E)$:

- $|V|$ is the number of vertices
- $|E|$ is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

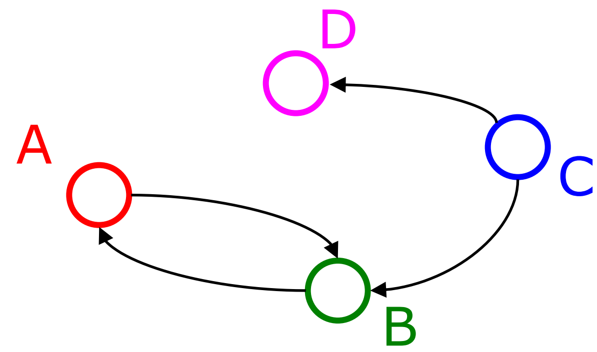


$V = \{A, B, C, D\}$
$E = \{(C, B), (A, B),$
$(B, A), (C, D)\}$

If $(u, v) \in E$, then v is a neighbor of u (i.e., v is adjacent to u)

- Order matters for directed edges:
  u is not adjacent to v unless $(v, u) \in E$

More Notation



For a graph $G = (V, E)$:

- $|V|$ is the number of vertices
- $|E|$ is the number of edges
  - Minimum? 0
  - Maximum for undirected? $|V||V+1|/2 \in O(|V|^2)$
  - Maximum for directed? $|V|^2 \in O(|V|^2)$

If $(u, v) \in E$, then v is a neighbor of u (i.e., v is adjacent to u)

- Order matters for directed edges:
  u is not adjacent to v unless $(v, u) \in E$

Examples Again

Which would use directed edges?

Which would have self-edges?

Which could have 0-degree nodes?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
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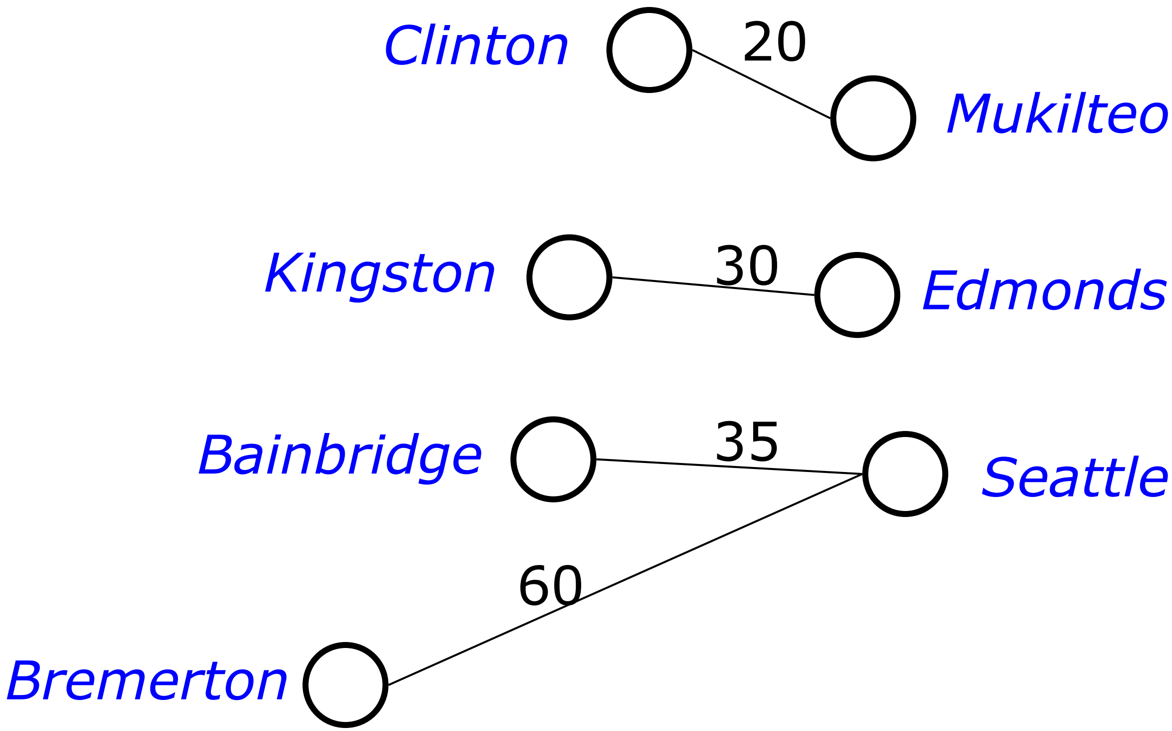
Weighted Graphs

In a weighted graph, each edge has a weight or cost

- Typically numeric (ints, decimals, doubles, etc.)
- Orthogonal to whether graph is directed
- Some graphs allow negative weights; many do not



Examples Again

What, if anything, might weights represent for each of these?

Do negative weights make sense?

- Web pages with links
- Facebook friends
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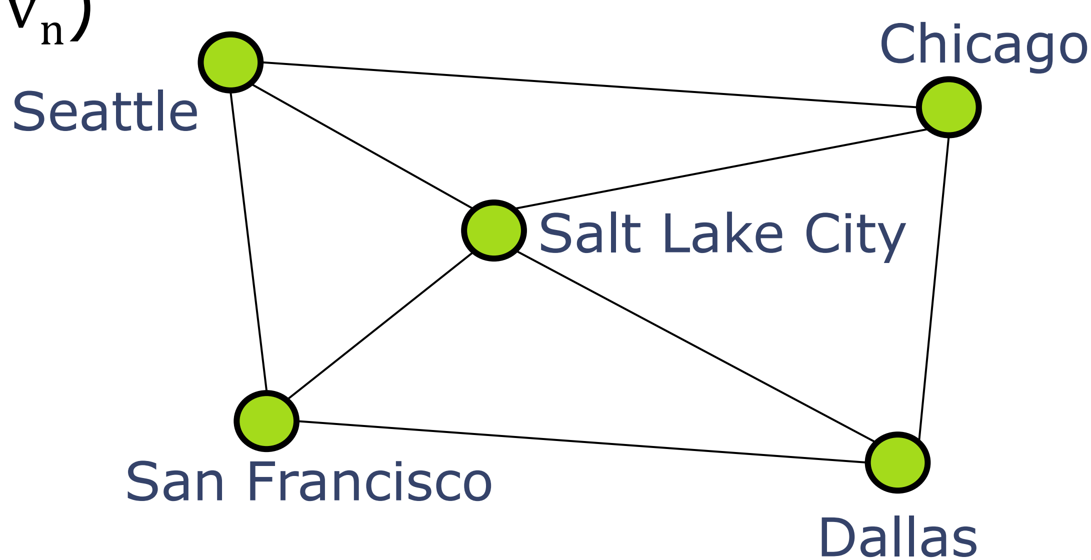
Paths and Cycles

We say "a path exists from v_0 to v_n" if there is a list of vertices $[v_0, v_1, \ldots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$.

A cycle is a path that begins and ends at the same node ($v_0 == v_n$)



Example path (that also happens to be a cycle):

[Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

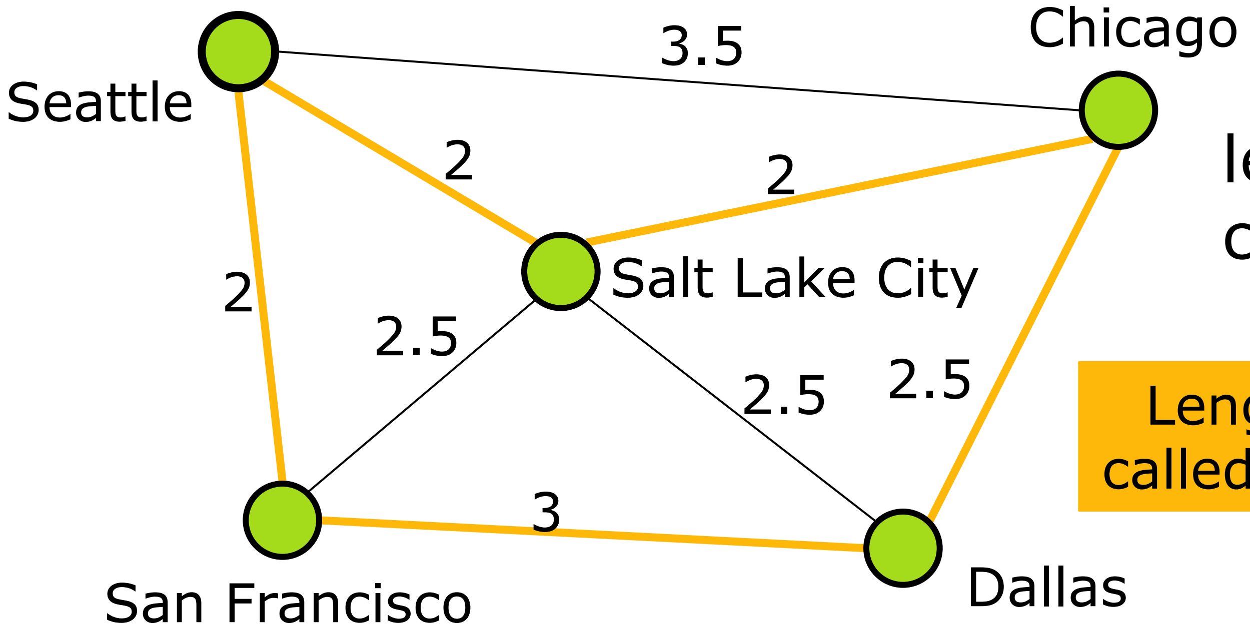
Path Length and Cost

Path length: Number of edges in a path

Path cost: Sum of the weights of each edge

Example where

P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]



length(**P**) = 5
cost(**P**) = 11.5

Length is sometimes called "unweighted cost"

Simple Paths and Cycles

A simple path repeats no vertices (except the first might be the last):

[Seattle, Salt Lake City, San Francisco, Dallas]

[Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

A cycle is a path that ends where it begins:

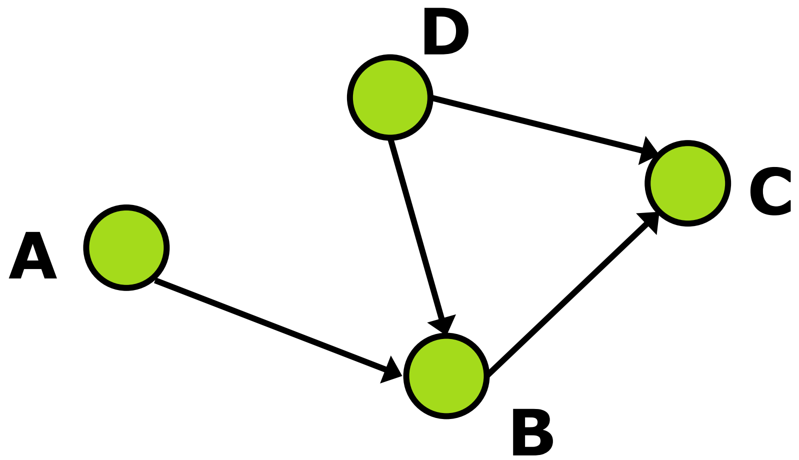
[Seattle, Salt Lake City, Seattle, Dallas, Seattle]

A simple cycle is a cycle and a simple path:

[Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

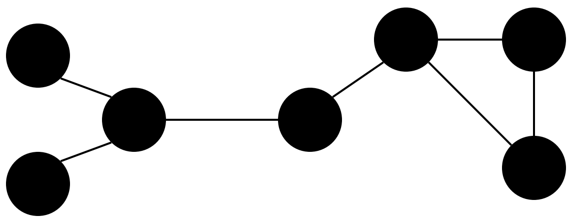
Example:



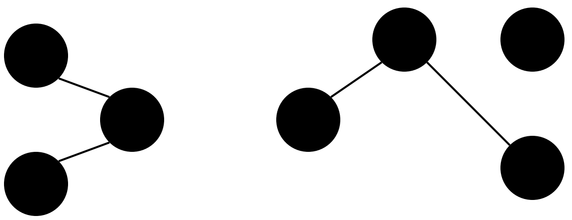
- Is there a path from A to D? No
- Does the graph contain any cycles? No

Undirected Graph Connectivity

An undirected graph is connected if for all pairs of vertices $u \neq v$, there exists a *path* from u to v

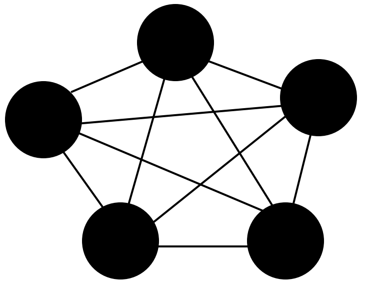


Connected graph



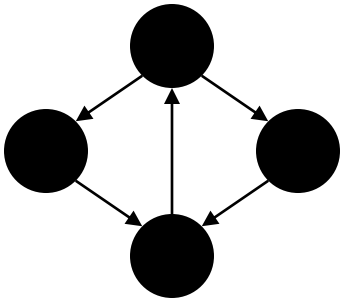
Disconnected graph

An undirected graph is complete, or fully connected, if for all pairs of vertices $u \neq v$ there exists an *edge* from u to v

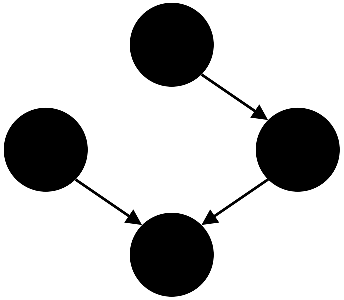


Directed Graph Connectivity

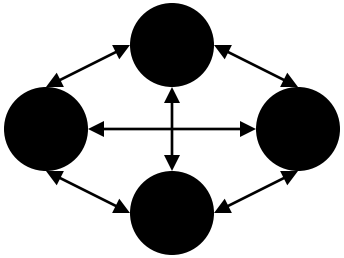
A directed graph is strongly connected if there is a path from every vertex to every other vertex



A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



A direct graph is complete or fully connected, if for all pairs of vertices $u \neq v$, there exists an *edge* from u to v



Examples Again

For undirected graphs: connected?
For directed graphs: strongly connected?
weakly connected?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
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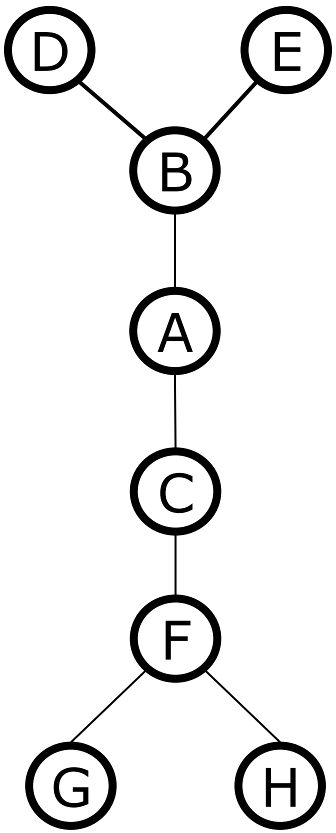
Trees as Graphs

When talking about graphs, we say a tree is a graph that is:

- undirected
- acyclic
- connected



All trees are graphs, but NOT all graphs are trees

How does this relate to the trees we know and "love"?

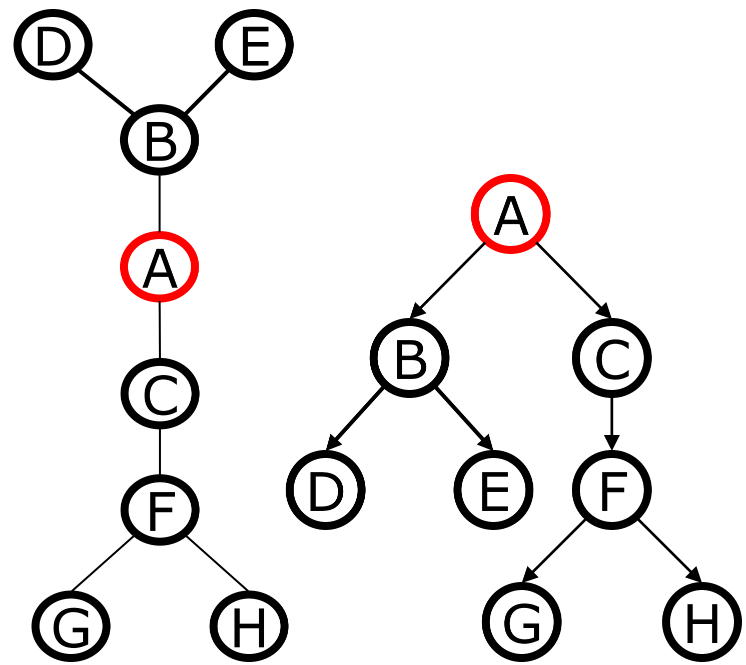
Rooted Trees

We are more accustomed to rooted trees where:

- We identify a unique root
- We think of edges as directed: parent to children

Picking a root gives a unique rooted tree

- The tree is simply drawn differently and with undirected edges



Rooted Trees

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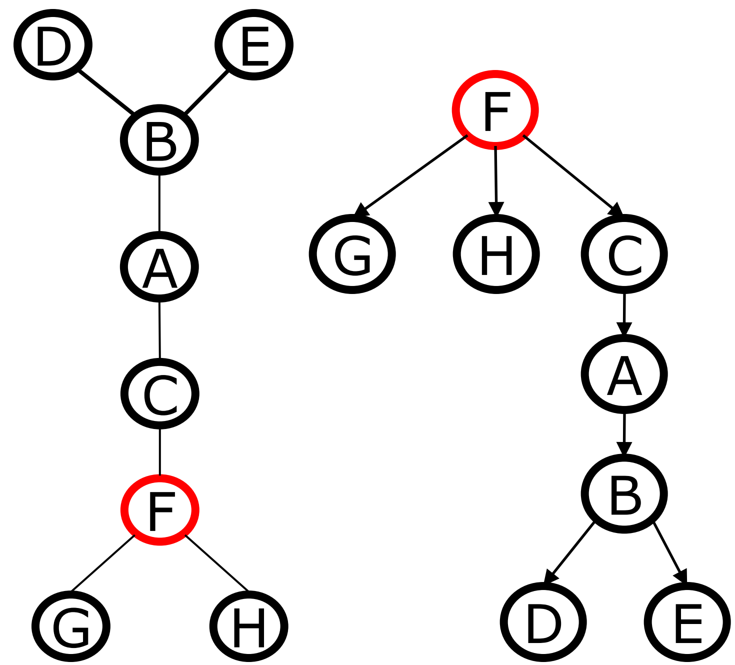
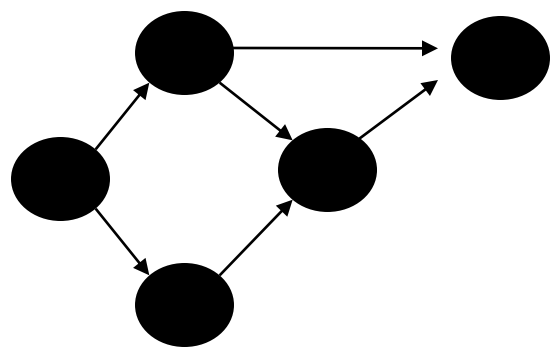


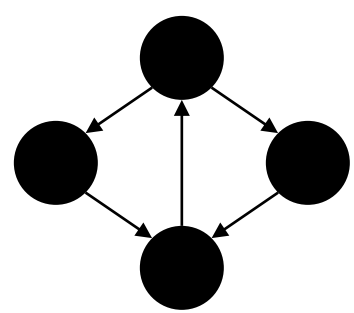
Directed Acyclic Graphs (DAGs)

A DAG is a directed graph with no directed cycles

- Every rooted directed tree is a DAG
- But not every DAG is a rooted directed tree



- Every DAG is a directed graph
- But not every directed graph is a DAG



Examples Again

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
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Density / Sparsity

Recall:

In an undirected graph, $0 \leq |E| < |V|^2$

Recall:

In a directed graph, $0 \leq |E| \leq |V|^2$

So for any graph, $|E|$ is $O(|V|^2)$

Another fact:

If an undirected graph is *connected*,
then $|E| \geq |V|-1$ (pigeonhole principle)

Density / Sparsity

$|E|$ is often much smaller than its maximum size

We do not always approximate as $|E|$ as $O(|V|^2)$

- This is a correct bound, but often not tight

If $|E|$ is $\Theta(|V|^2)$ (the bound is tight), we say the graph is dense

- More sloppily, dense means "lots of edges"

If $|E|$ is $O(|V|)$ we say the graph is sparse

- More sloppily, sparse means "most possible edges missing"

Insert humorous statement here

GRAPH DATA STRUCTURES

What's the Data Structure?

Graphs are often useful for lots of data and questions

- Example: "What's the lowest-cost path from x to y"

But we need a data structure that represents graphs

Which data structure is "best" can depend on:

- properties of the graph (e.g., dense versus sparse)
- the common queries about the graph ("is (u,v) an edge?" vs "what are the neighbors of node u?")

We will discuss two standard graph representations

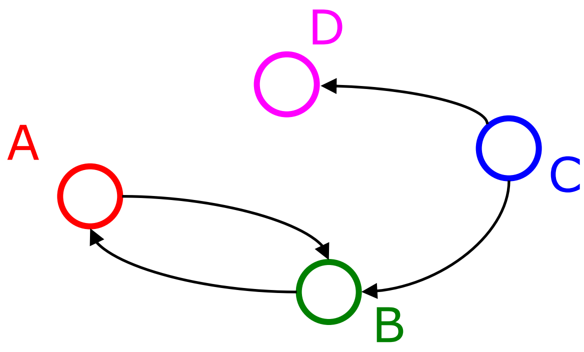
- Adjacency Matrix and Adjacency List
- Different trade-offs, particularly time versus space

Adjacency Matrix

Assign each node a number from 0 to $|V|$-1

A |V| x |V| matrix of Booleans (or 0 vs. 1)

- Then $\mathrm{M[u][v]} == \mathrm{true}$ means there is an edge from u to v



	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

Running time to:

- Get a vertex's out-edges:
- Get a vertex's in-edges:
- Decide if some edge exists:
- Insert an edge:
- Delete an edge:

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Space requirements:

Best for sparse or dense graphs?

Adjacency Matrix Properties

Running time to:

- Get a vertex's out-edges: $O(|V|)$
- Get a vertex's in-edges: $O(|V|)$
- Decide if some edge exists: $O(1)$
- Insert an edge: $O(1)$
- Delete an edge: $O(1)$

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Space requirements:

$O(|V|^2)$

Best for sparse or dense graphs? *dense*

Adjacency Matrix Properties

How will the adjacency matrix vary for an undirected graph?

- Will be symmetric about diagonal axis
- Matrix: Could we save space by using only about half the array?

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	*F*	*T*	*F*	T
D	*F*	*F*	*T*	*F*

- But how would you "get all neighbors"?

Adjacency Matrix Properties

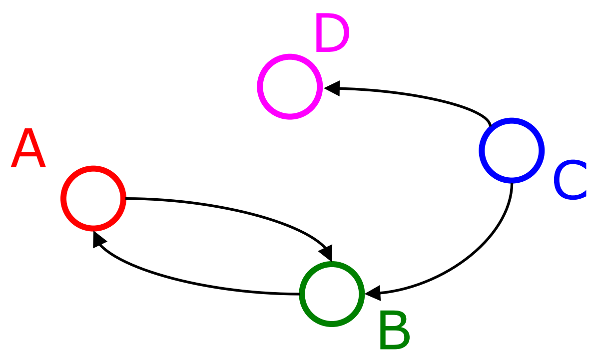
How can we adapt the representation for weighted graphs?

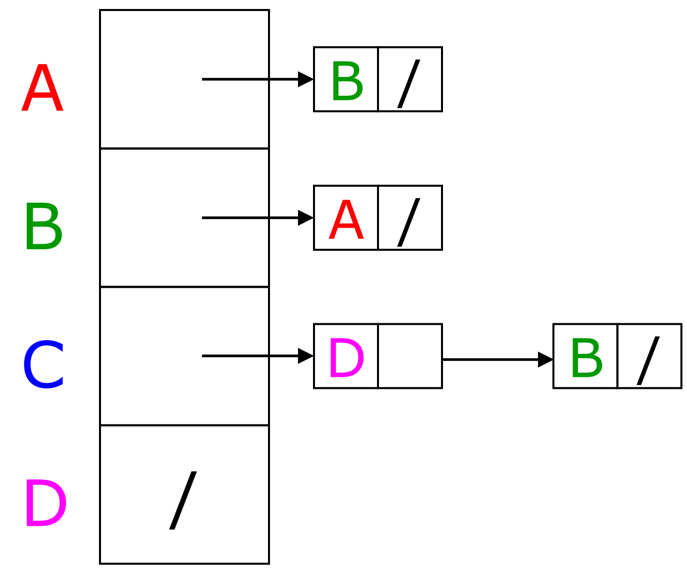
- Instead of Boolean, store a number in each cell
- Need some value to represent ‘not an edge’
  - 0, -1, or some other value based on how you are using the graph
  - Might need to be a separate field if no restrictions on weights

Adjacency List

Assign each node a number from 0 to $|V|$-1

- An array of length $|V|$ in which each entry stores a list of all adjacent vertices (e.g., linked list)

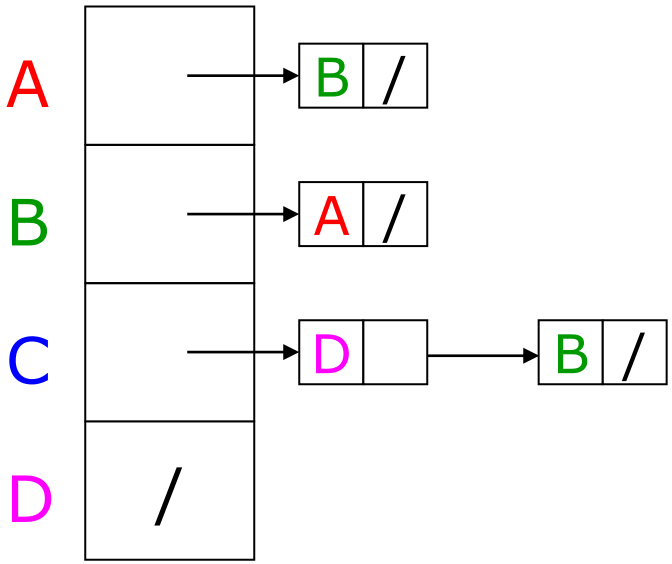




Adjacency List Properties

Running time to:

- Get a vertex's out-edges:
- Get a vertex's in-edges:
- Decide if some edge exists:
- Insert an edge:
- Delete an edge:



Space requirements:

Best for sparse or dense graphs?

Adjacency List Properties

Running time to:

- Get a vertex's out-edges:
  $O(d)$ where d is out-degree of vertex
- Get a vertex's in-edges:
  $O(|E|)$ (could keep a second adjacency list for this!)
- Decide if some edge exists:
  $O(d)$ where d is out-degree of source
- Insert an edge:
  $O(1)$ (unless you need to check if it's already there)
- Delete an edge:
  $O(d)$ where d is out-degree of source

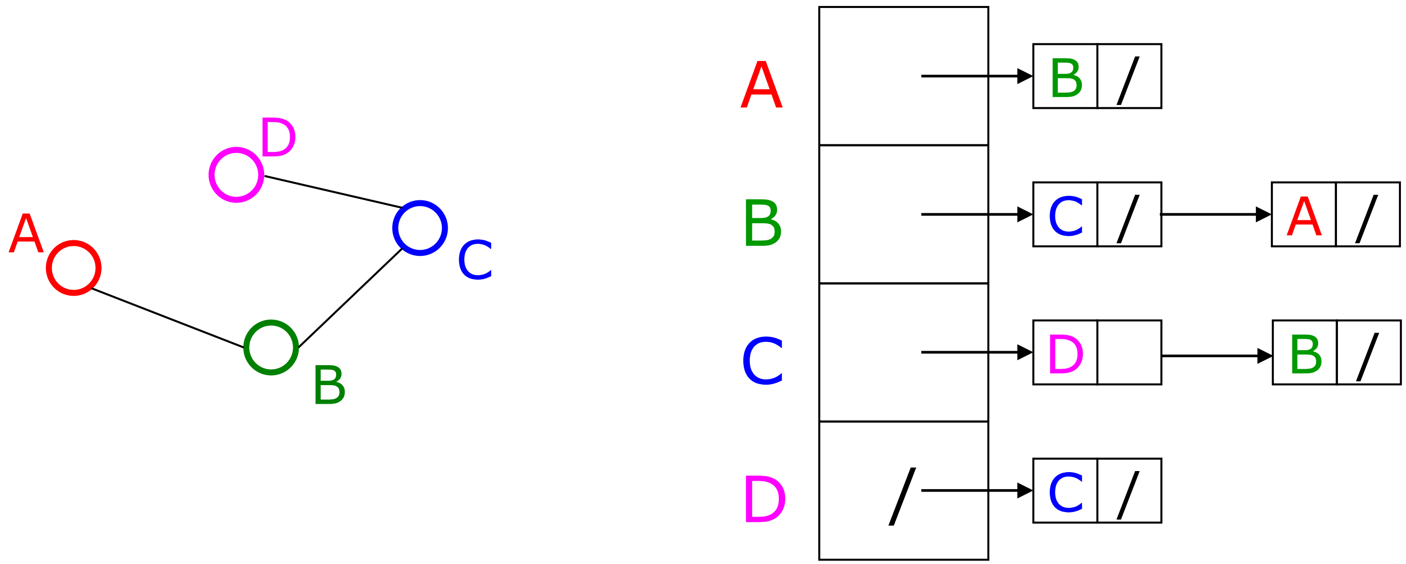
Space requirements: $O(|V|+|E|)$

Best for sparse or dense graphs? *sparse*

Undirected Graphs

Adjacency lists also work well for undirected graphs with one caveat

- Put each edge in two lists to support efficient "get all neighbors"



Which is better?

Graphs are often sparse

- Streets form grids
- Airlines rarely fly to all cities

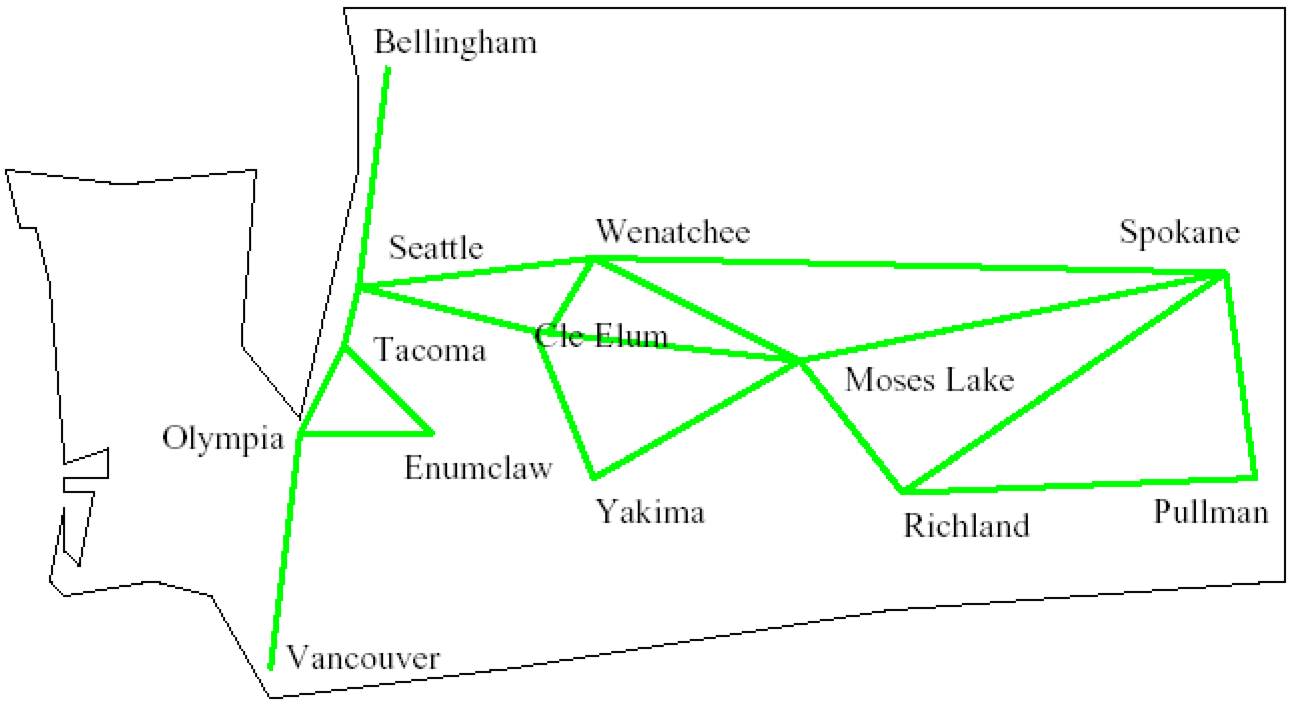
Adjacency lists should generally be your default choice

- Slower performance compensated by greater space savings

Might be easier to list what isn't a graph application…

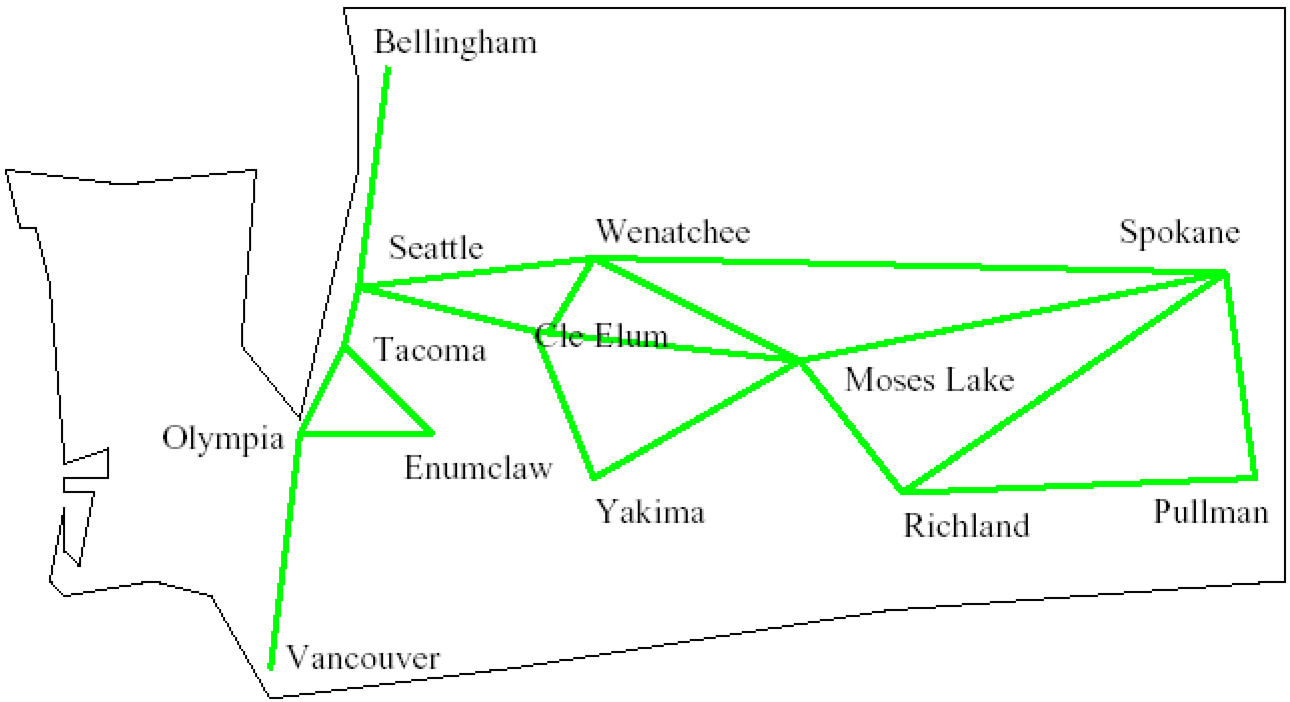
APPLICATIONS OF GRAPHS: TRAVERSALS

Application: Moving Around WA State



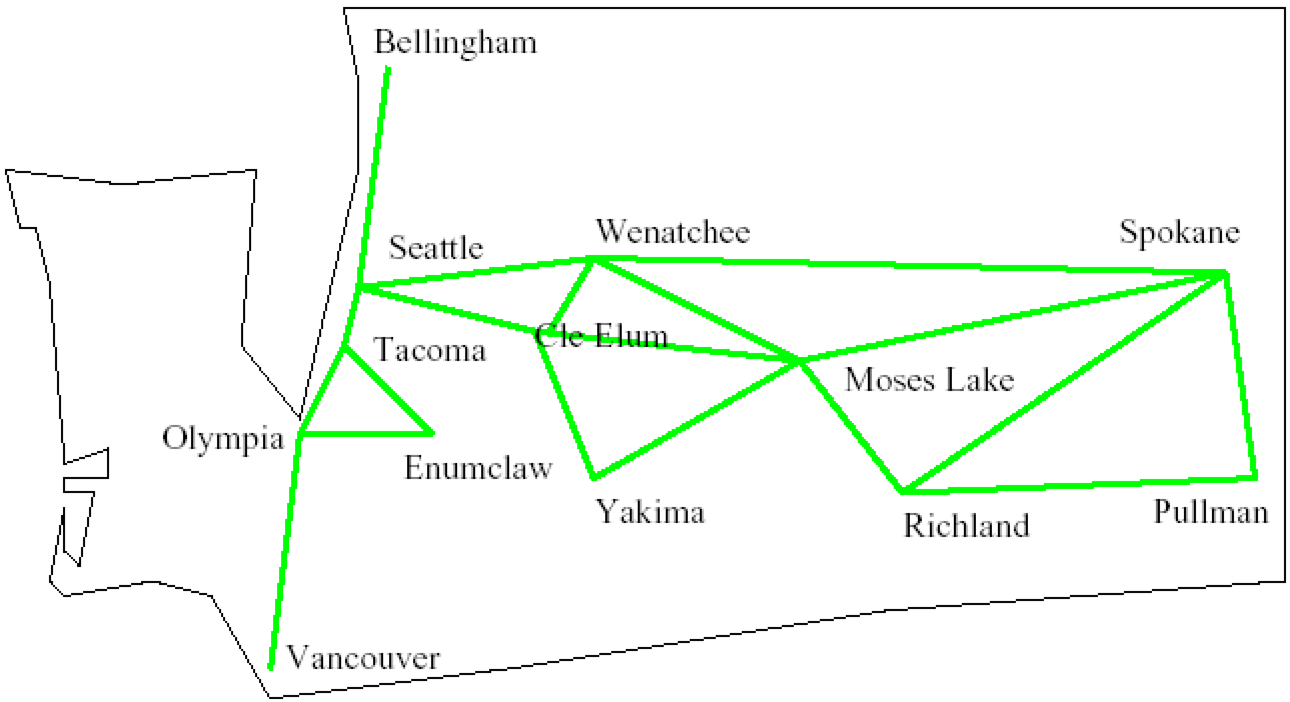
What's the *shortest way* to get from Seattle to Pullman?

Application: Moving Around WA State



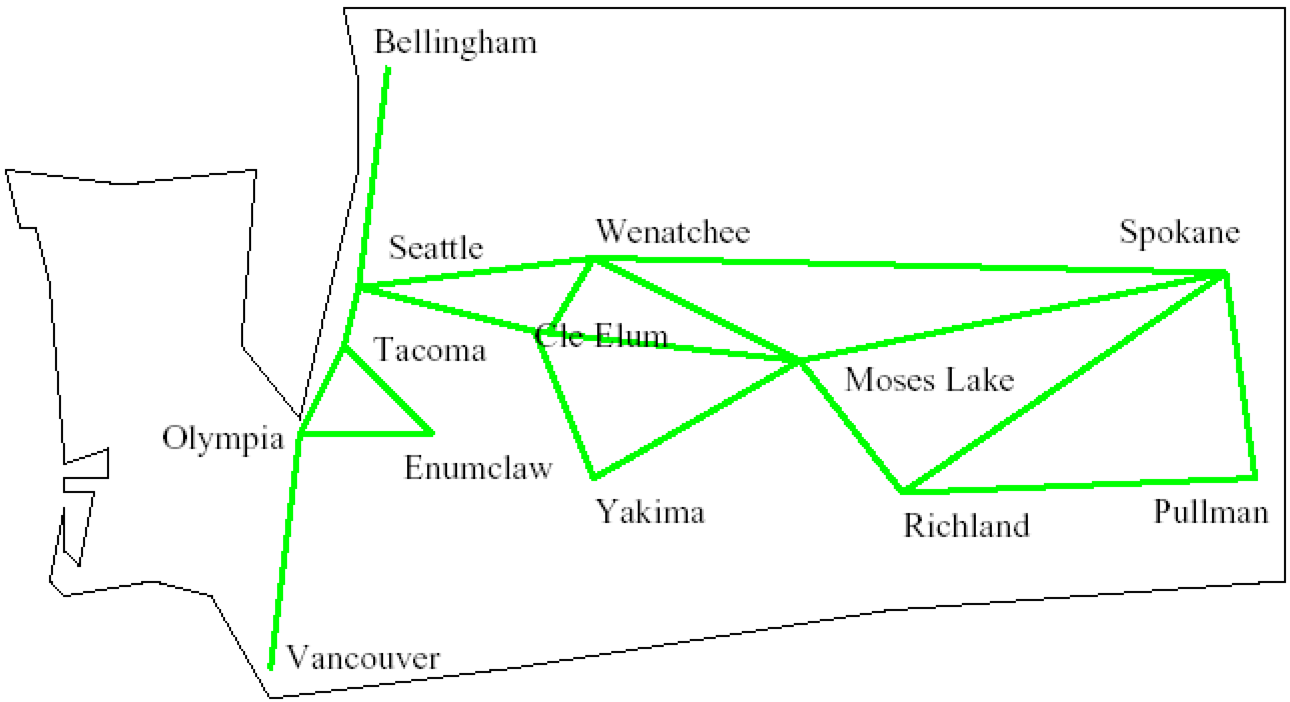
What's the *fastest way* to get from Seattle to Pullman?

Application: Reliability of Communication



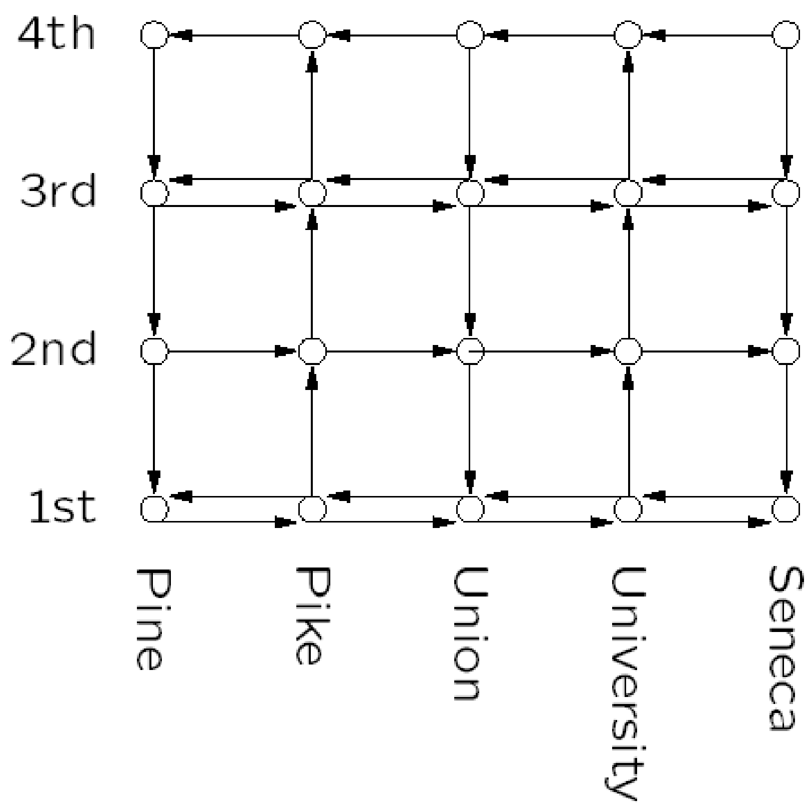
If Wenatchee's phone exchange *goes down*, can Seattle still talk to Pullman?

Application: Reliability of Communication



If Tacomas's phone exchange *goes down*, can Olympia still talk to Spokane?

Applications: Bus Routes Downtown



If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

Graph Traversals

For an arbitrary graph and a starting node v, find all nodes reachable from v (i.e., there exists a path)

- Possibly "do something" for each node (print to output, set some field, return from iterator, etc.)

Related Problems:

- Is an undirected graph connected?
- Is a digraph weakly/strongly connected?
  - For strongly, need a cycle back to starting node

Graph Traversals

Basic Algorithm for Traversals:

- Select a starting node
- Make a set of nodes adjacent to current node
- Visit each node in the set but "mark" each nodes after visiting them so you don't revisit them (and eventually stop)
- Repeat above but skip "marked nodes"

In Rough Code Form

```
traverseGraph(Node start) {
    Set pending = emptySet();
    pending.add(start)
    mark start as visited
    while(pending is not empty) {
        next = pending.remove()
        for each node u adjacent to next
            if(u is not marked) {
                mark u
                pending.add(u)
            }
        }
    }
}
```

Running Time and Options

Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$ if using an adjacency list

The order we traverse depends entirely on how add and remove work/are implemented

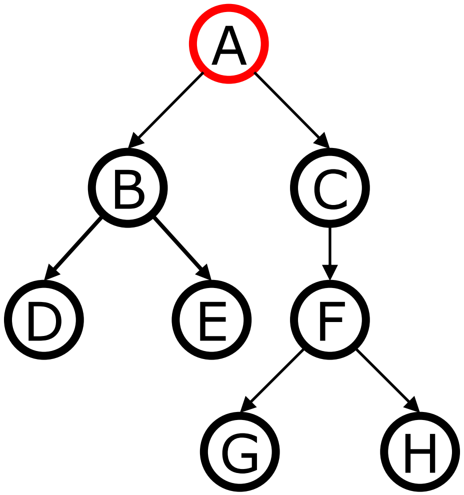
- DFS: a stack "depth-first graph search"
- BFS: a queue "breadth-first graph search"

DFS and BFS are "big ideas" in computer science

- Depth: recursively explore one part before going back to the other parts not yet explored
- Breadth: Explore areas closer to start node first

Recursive DFS, Example with Tree

A tree is a graph and DFS and BFS are particularly easy to "see" in one

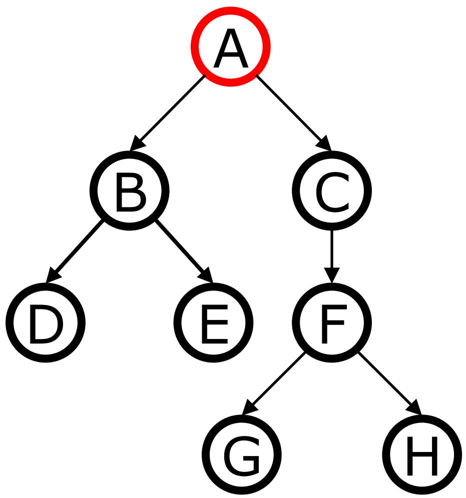



```
DFS(Node start) {
    mark and process start
    for each node u adjacent to start
        if u is not marked
            DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

- This is a "pre-order traversal" for trees
- The marking is unneeded here but because we support arbitrary graphs, we need a means to process each node exactly once

DFS with Stack, Example with Tree

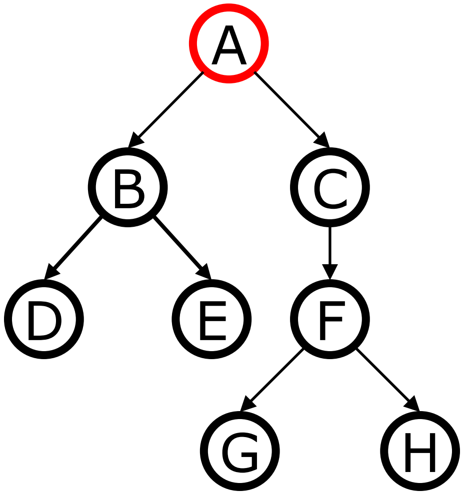



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
     next = s.pop() // and "process"
     for each node u adjacent to next
        if(u is not marked)
           mark u and push onto s
  }
}
```

Order processed: A, C, F, H, G, B, E, D

- A different order but still a perfectly fine traversal of the graph

BFS with Queue, Example with Tree




```
BFS(Node start) {
   initialize queue q to hold start
   mark start as visited
   while(q is not empty) {
      next = q.dequeue() // and "process"
      for each node u adjacent to next
         if(u is not marked)
            mark u and enqueue onto q
   }
}
```

Order processed: A, B, C, D, E, F, G, H

- A "level-order" traversal

DFS/BFS Comparison

BFS always finds the shortest path (or "optimal solution") from the starting node to a target node

- Storage for BFS can be extremely large
- A k-nary tree of height h could result in a queue size of k^h

DFS can use less space in finding a path

- If longest path in the graph is p and highest out-degree is d then DFS stack never has more than $d \cdot p$ elements

Implications

For large graphs, DFS is hugely more memory efficient, *if we can limit the maximum path length to some fixed d.*

If we *knew* the distance from the start to the goal in advance, we could simply *not add any children to stack after level d*

But what if we don't know *d* in advance?

Iterative Deepening (IDFS)

Algorithms

- Try DFS up to recursion of K levels deep.
- If fails, increment K and start the entire search over

Performance:

- Like BFS, IDFS finds shortest paths
- Like DFS, IDFS uses less space
- Some work is repeated but minor compared to space savings

Saving the Path

Our graph traversals can answer the standard *reachability* question:

"Is there a path from node x to node y?"

But what if we want to actually output the path?

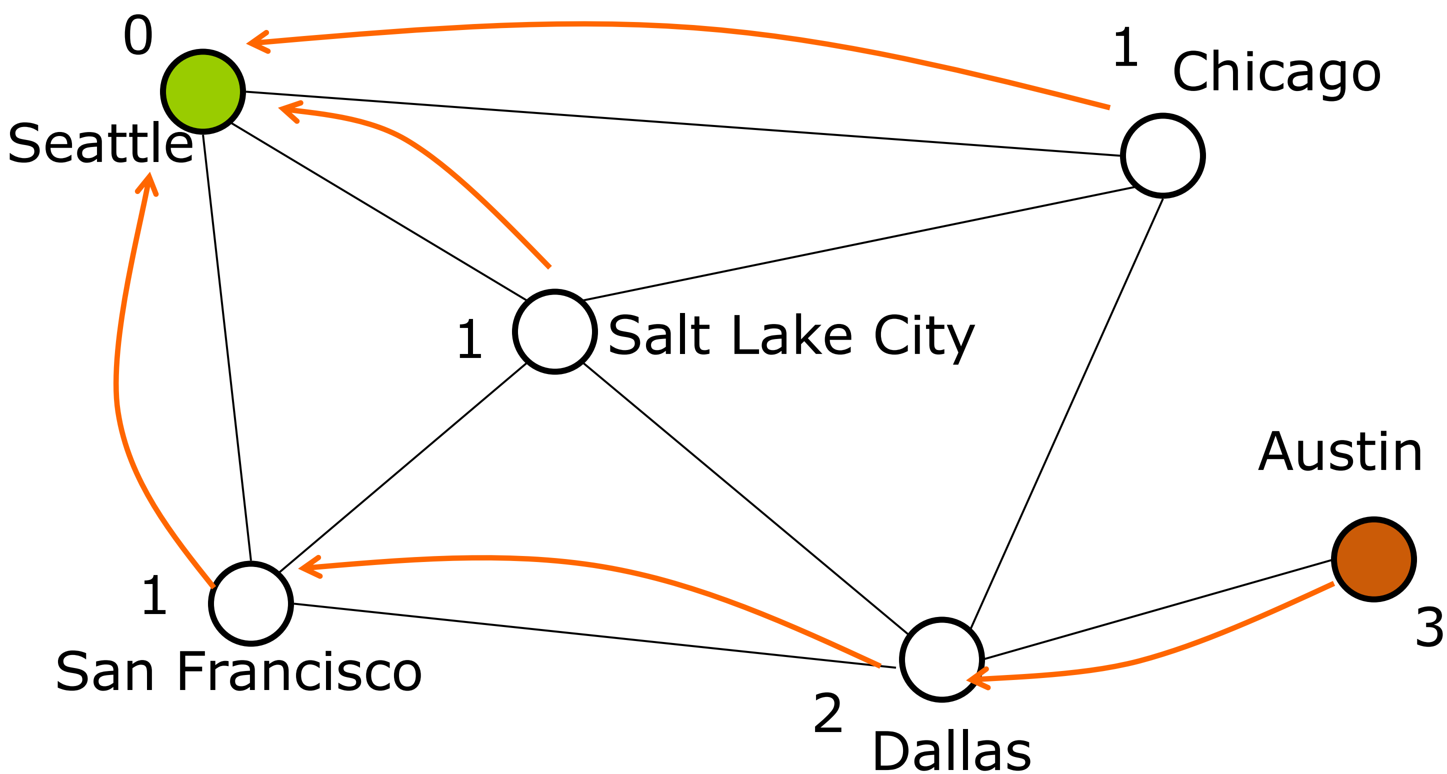
Easy:

- Store the previous node along the path:
  When processing *u* causes us to add *v* to the search, set *v.path* field to be *u*)
- When you reach the goal, follow path fields back to where you started (and then reverse the answer)
- What's an easy way to do the reversal? A Stack!!

Example using BFS

What is a path from Seattle to Austin?

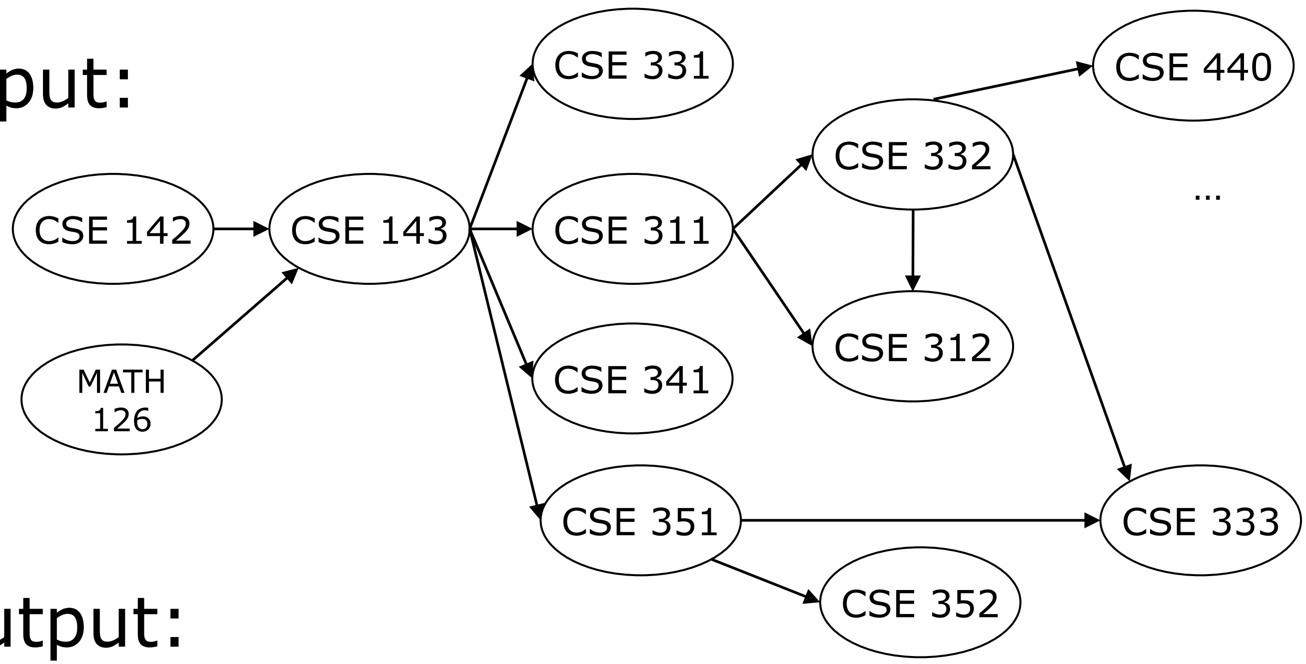
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



Topological Sort

Problem: Given a DAG $G=(V, E)$, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:

- 142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352

Disclaimer: Do not use for official advising purposes!
(Implies that CSE 332 is a pre-req for CSE 312 – not true)

Questions and Comments

Terminology:

A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Why do we perform topological sorts only on DAGs?

- Because a cycle means there is no correct answer

Is there always a unique answer?

- No, there can be one or more answers depending on the provided graph

What DAGs have exactly 1 answer?

- Lists

Uses Topological Sort

Figuring out how to finish your degree

Computing the order in which to recalculate cells in a spreadsheet

Determining the order to compile files with dependencies

In general, use a dependency graph to find an allowed order of execution

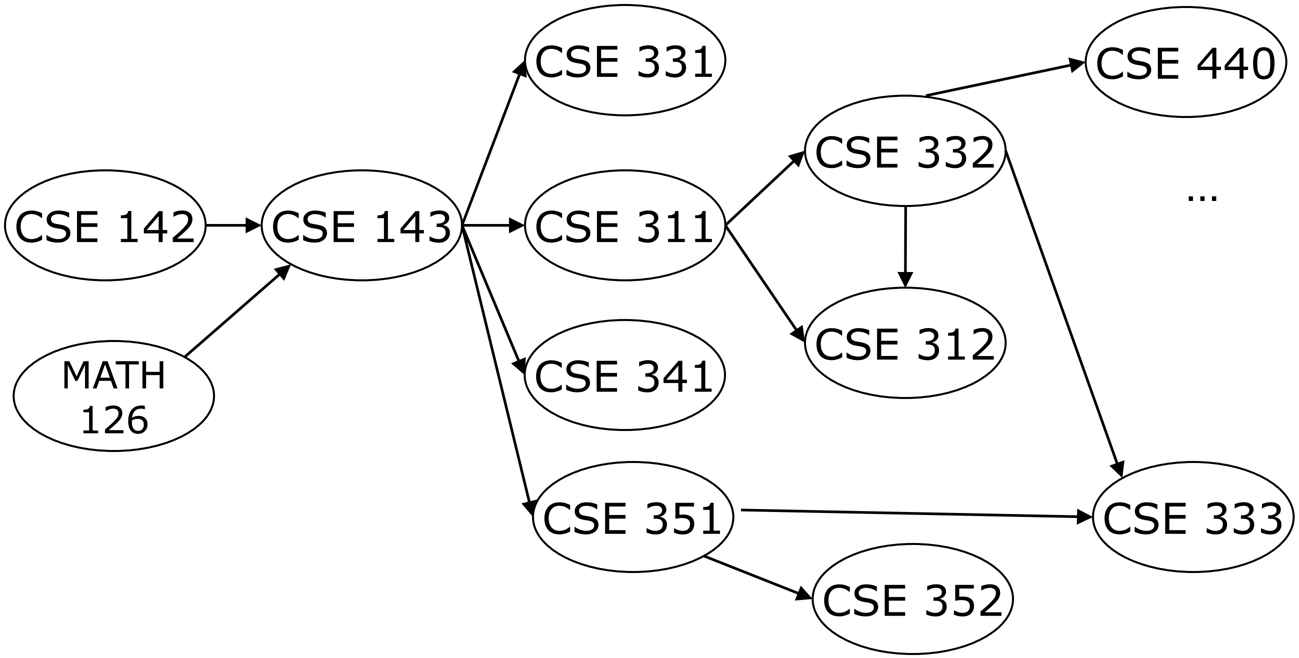
Topological Sort: First Approach

1. Label each vertex with its in-degree
   - Think "write in a field in the vertex"
   - You could also do this with a data structure on the side

2. While there are vertices not yet outputted:
   a) Choose a vertex **v** labeled with in-degree of 0
   b) Output **v** and "remove it" from the graph
   c) For each vertex **u** adjacent to **v**, decrement in-degree of **u**
      - (i.e., **u** such that (**v**,**u**) is in **E**)

Example

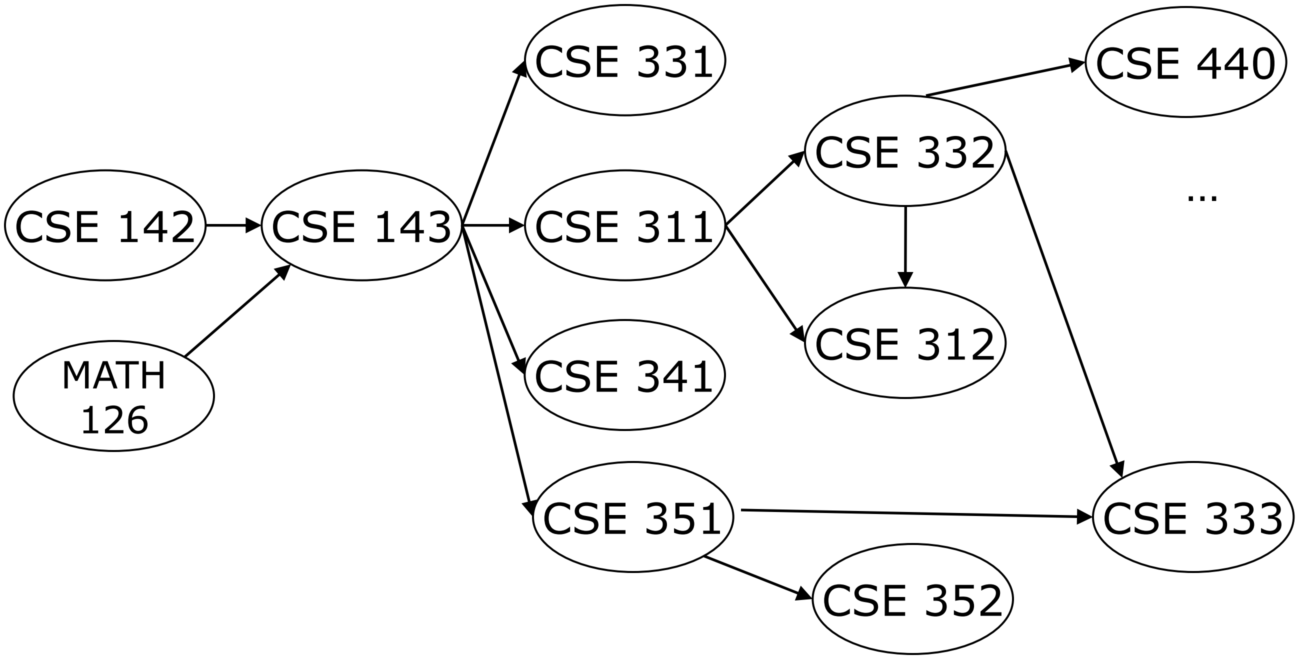
Output:




```
Node:    126 142 143 311 312 331 332 333 341 351 352 440
Removed?
In-deg:
```

Example

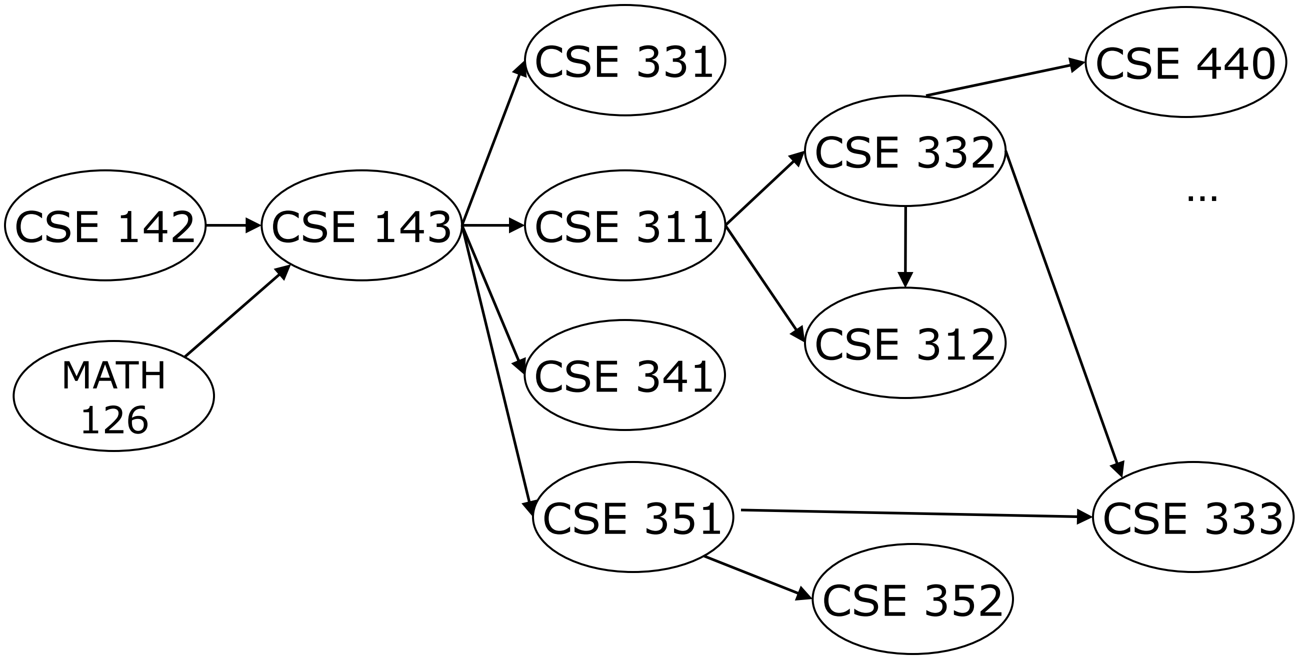
Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1

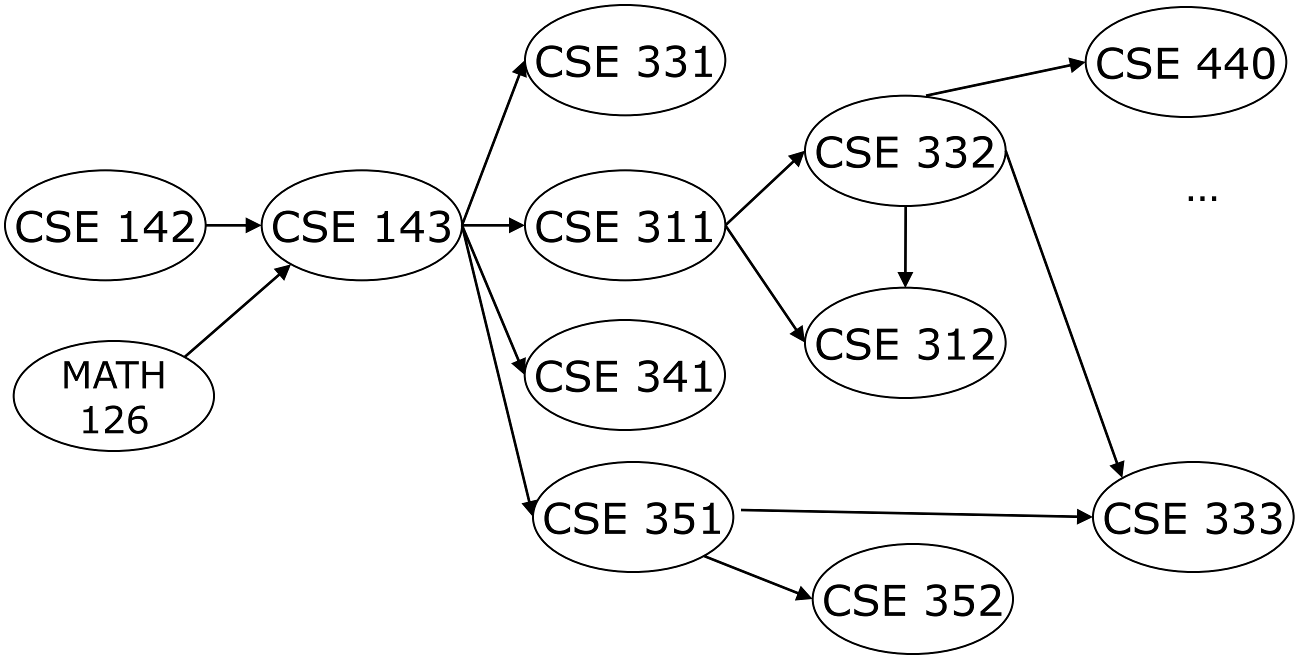
Example

Output:
126



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1									

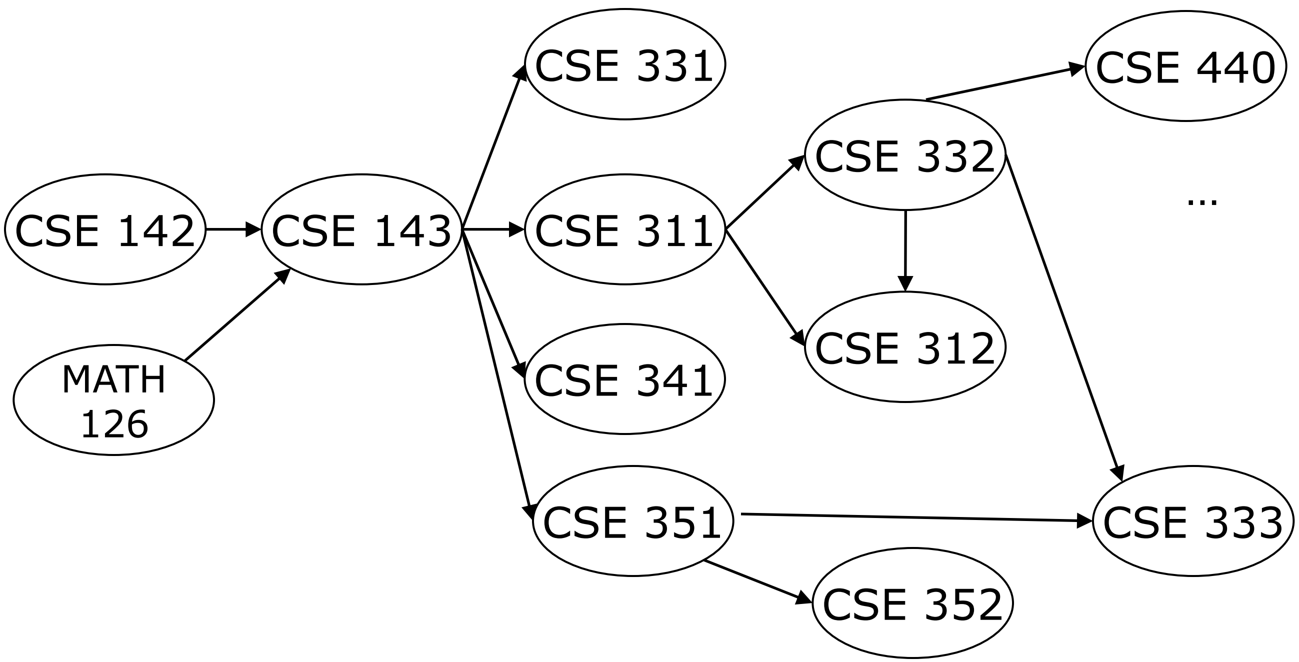
Example



Output:
126
142

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

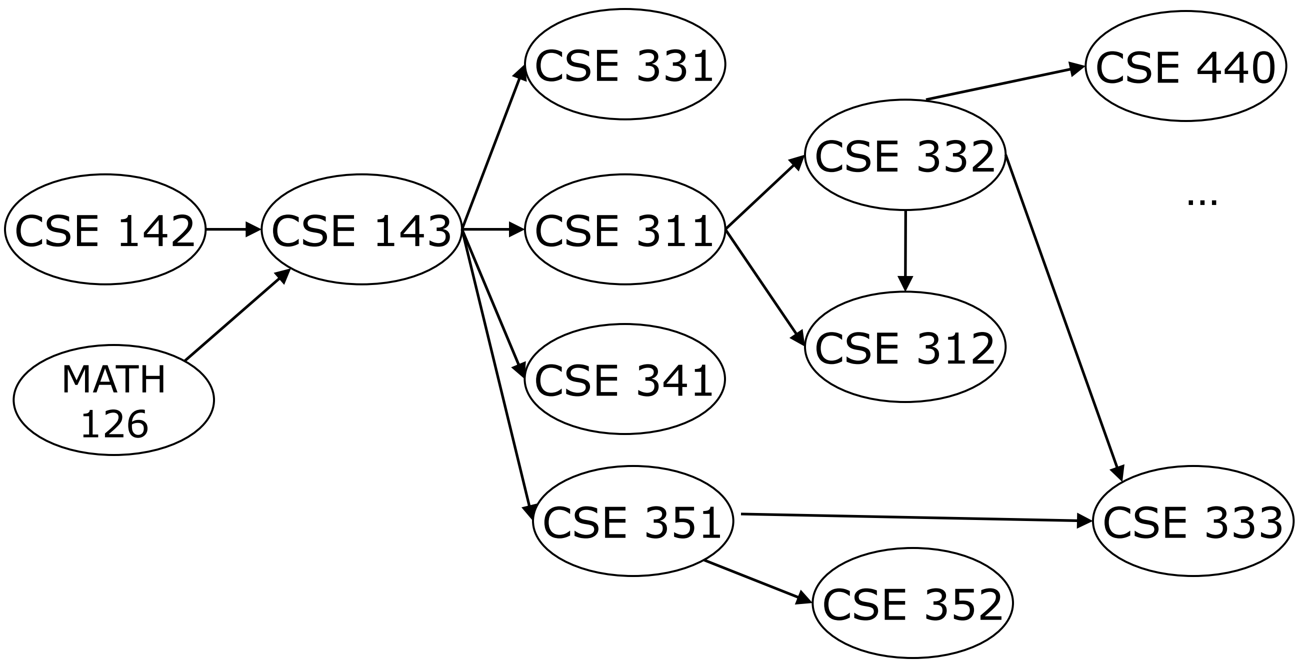
Example



Output:
126
142
143

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x									
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

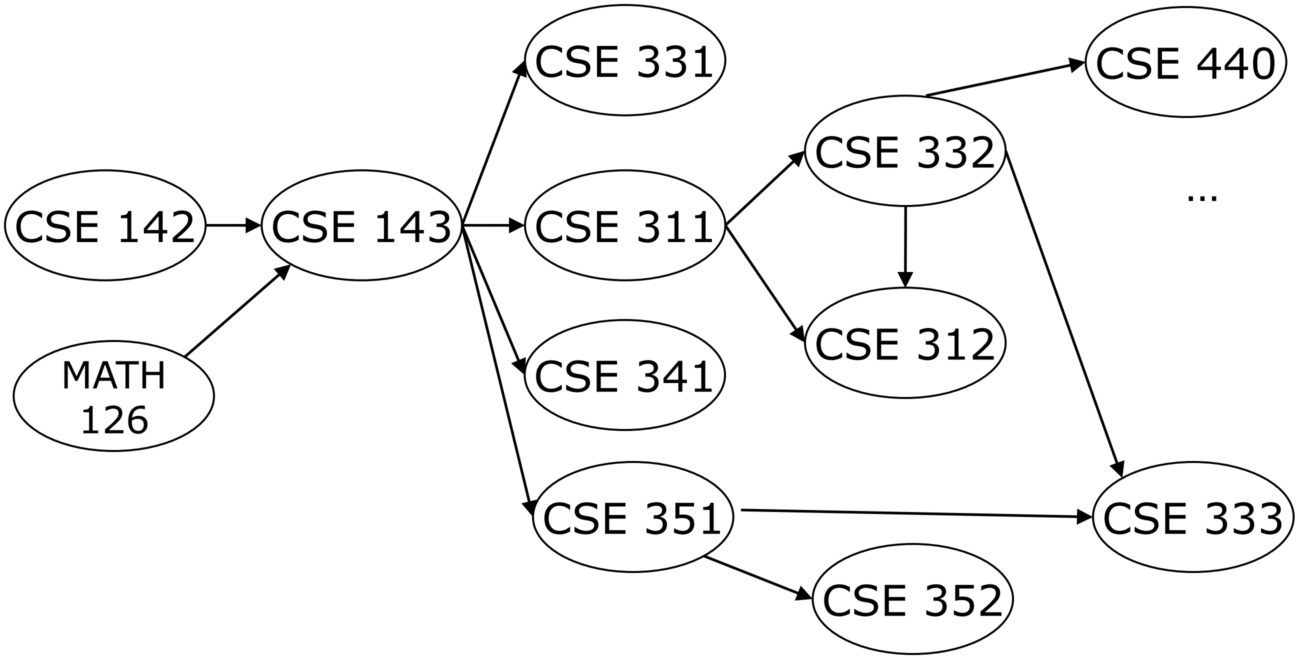
Example



Output:
126
142
143
311

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x								
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

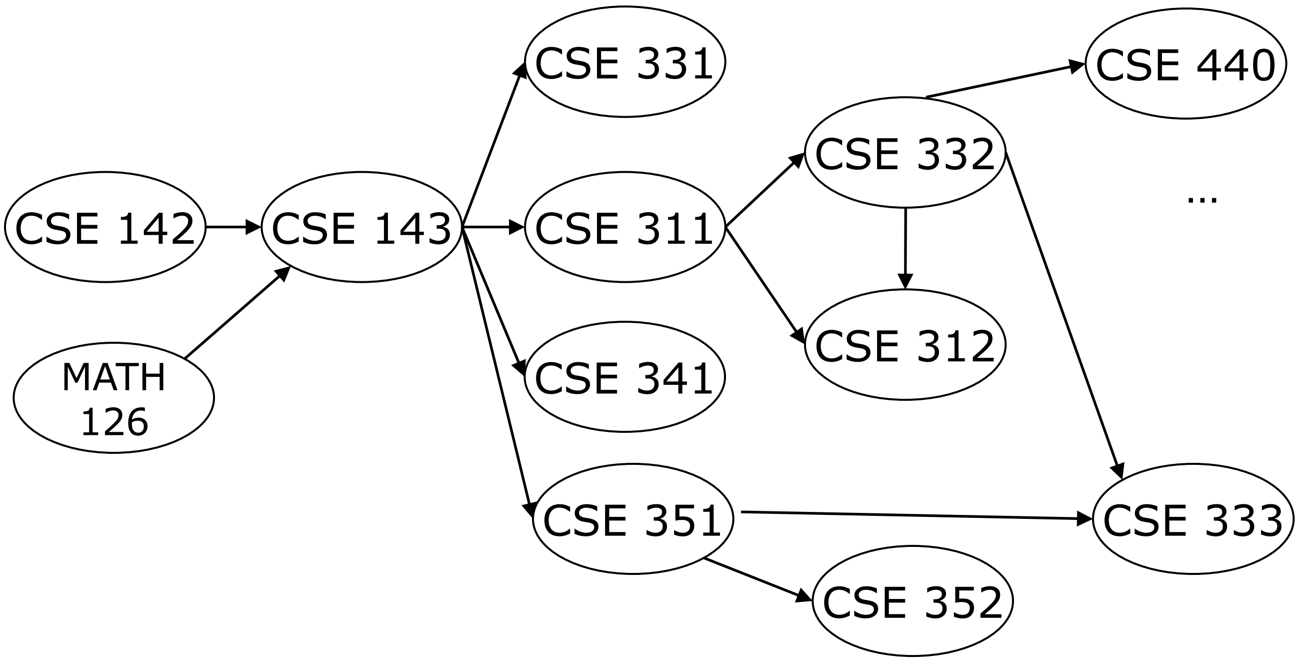
Example



Output:
126
142
143
311
331

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x						
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

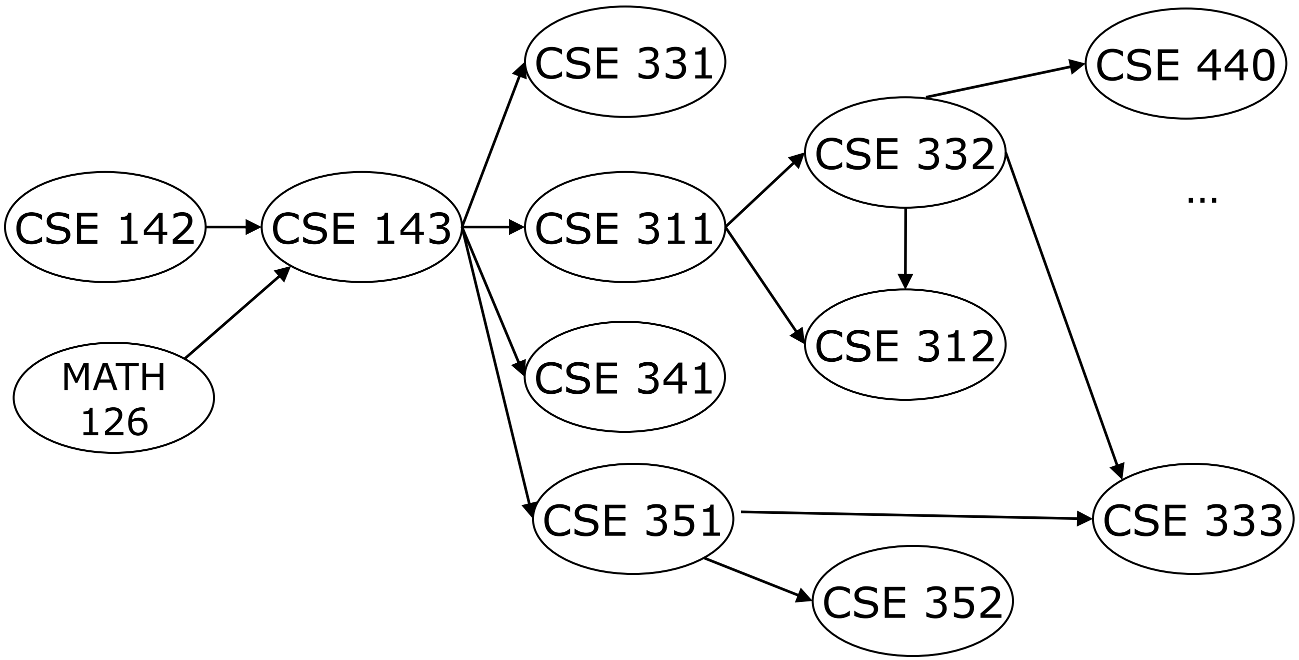
Example



Output:
126
142
143
311
331
332

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x	x					
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

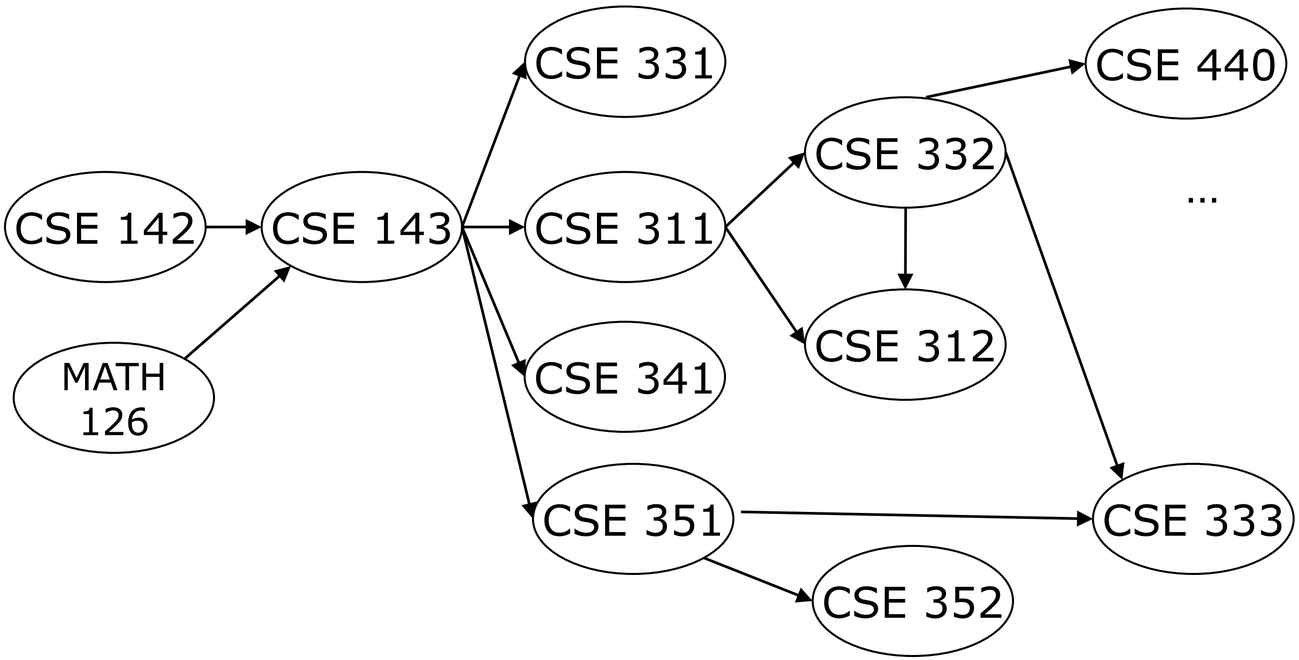
Example



Output:
126
142
143
311
331
332
312

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x					
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

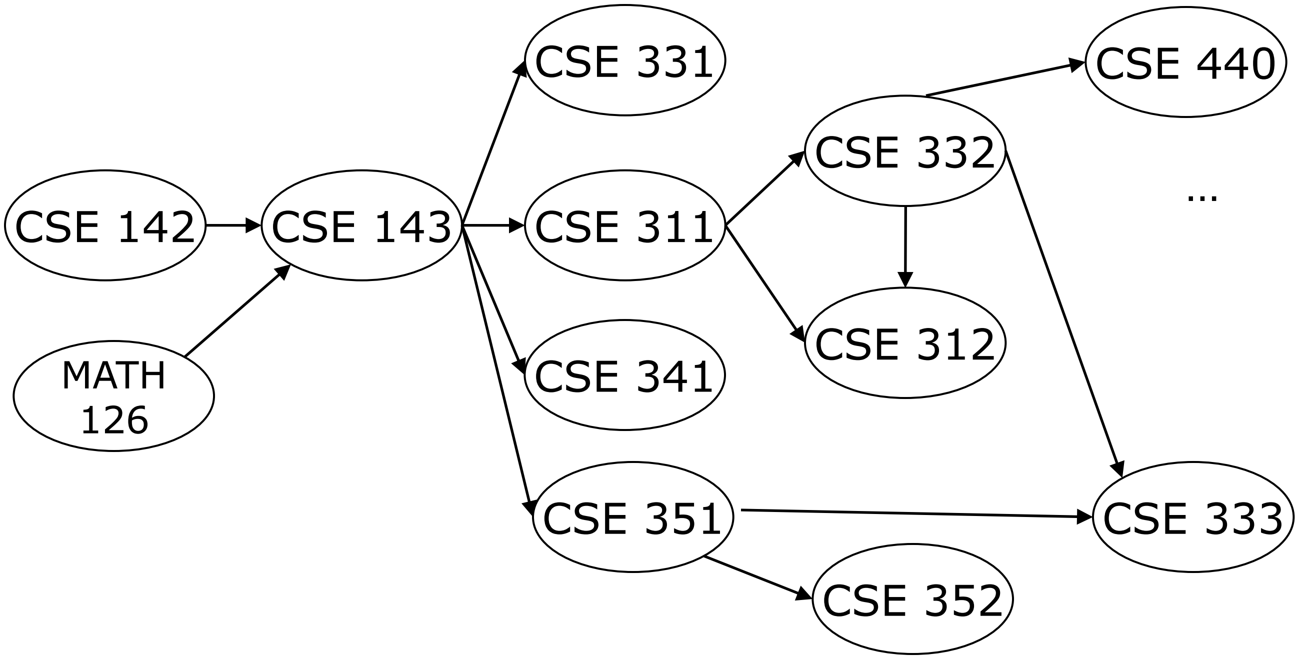
Example



Output:
126
142
143
311
331
332
312
341

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x			
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

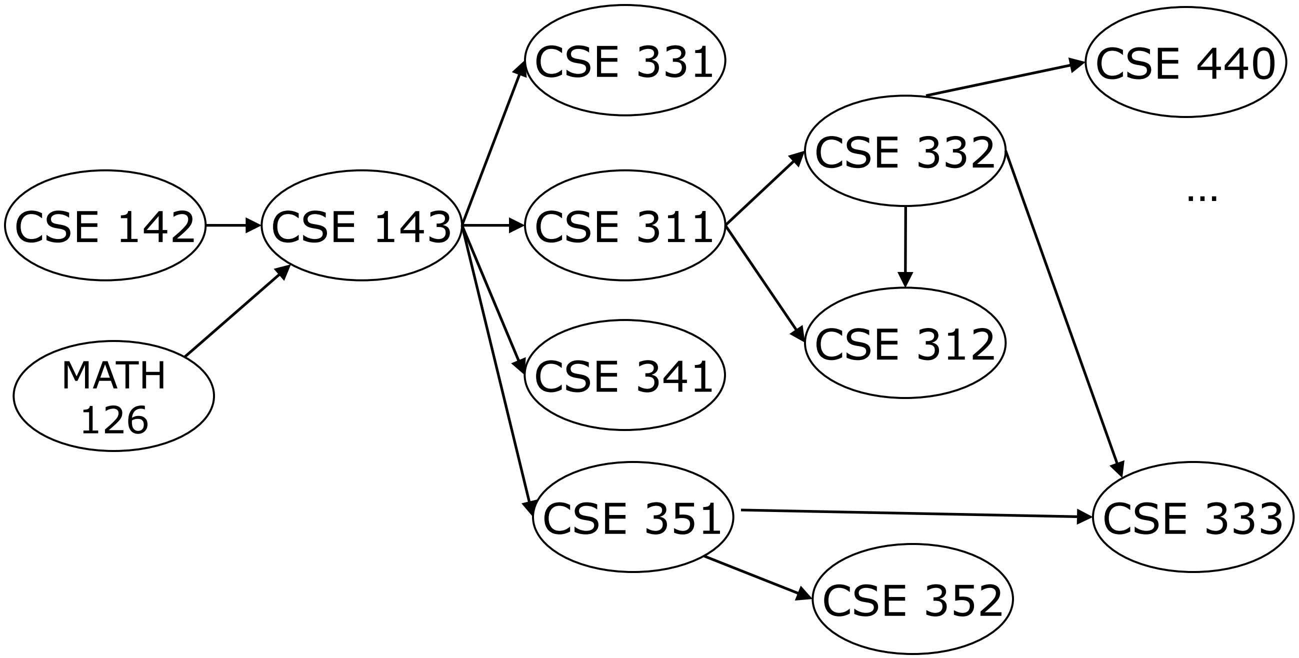
Example



Output:
126
142
143
311
331
332
312
341
351

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x	x		
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



Output:
126
142
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332
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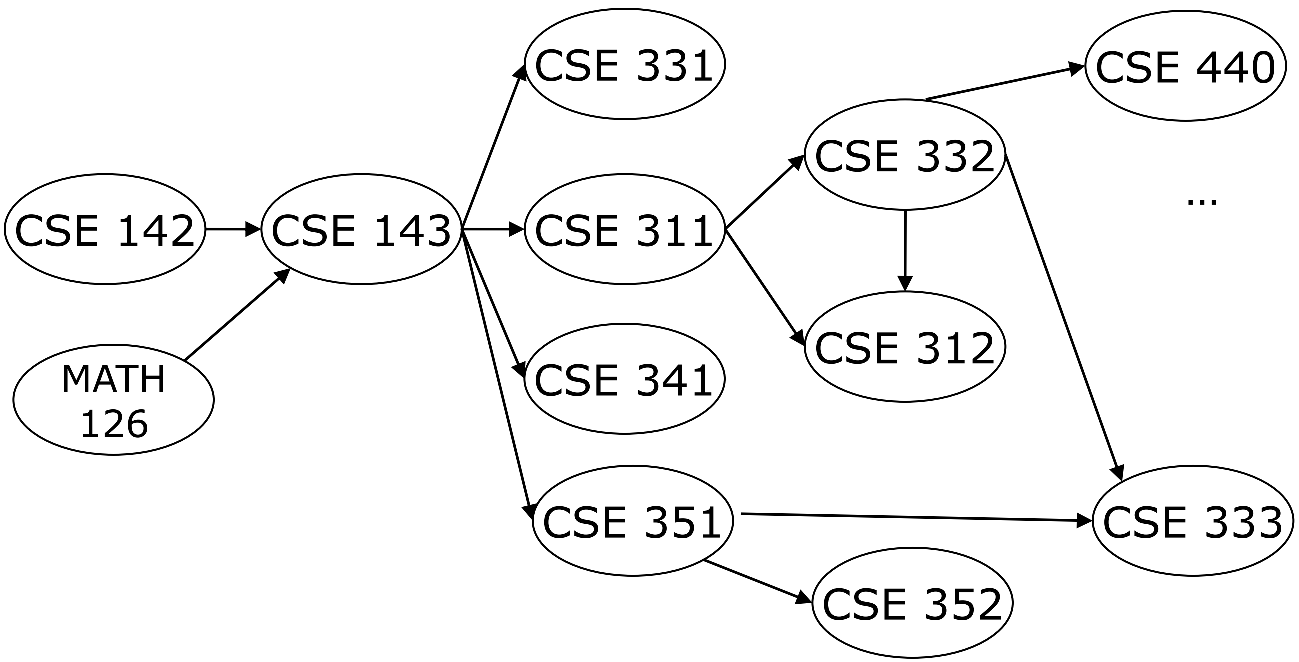
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x		
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example

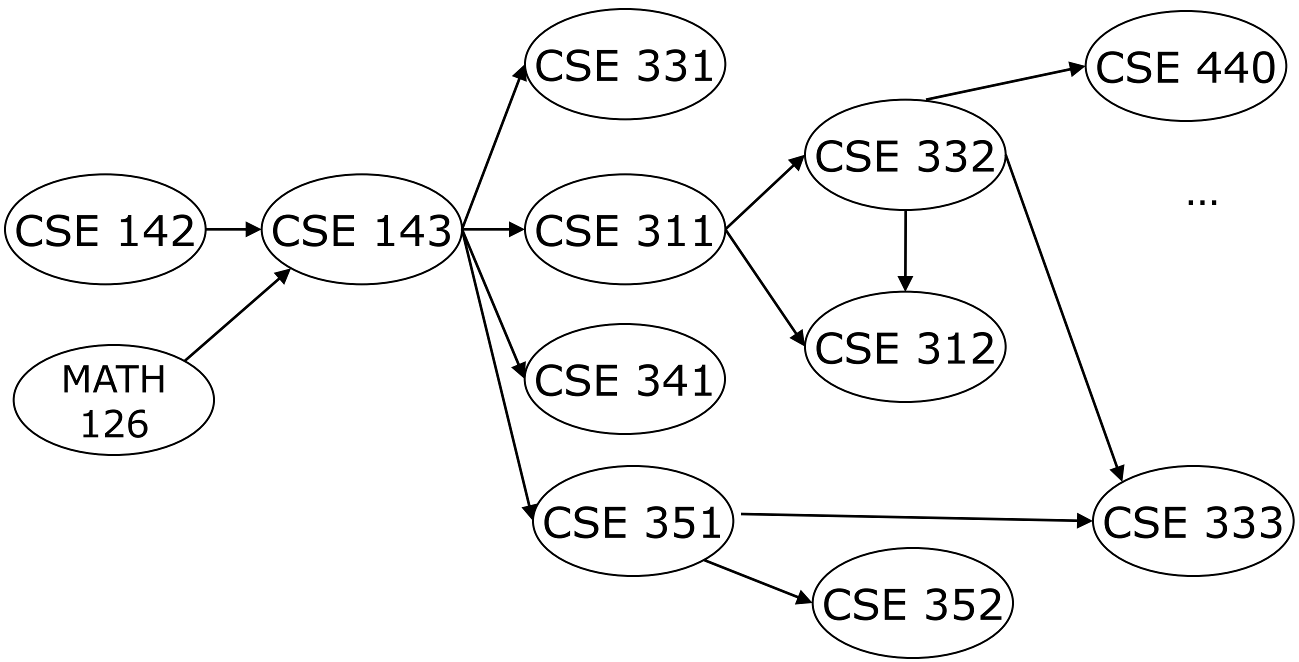


Output:

126 352
142
143
311
331
332
312
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



Output:

126 352
142 440
143
311
331
332
312
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Running Time?

```
labelEachVertexWithItsInDegree();

for(i=0; i < numVertices; i++) {
      v = findNewVertexOfDegreeZero();
      put v next in output
      for each w adjacent to v
         w.indegree--;
}
```

What is the worst-case running time?

- Initialization $O(|V| + |E|)$ (assuming adjacency list)
- Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
- Sum of all decrements $O(|E|)$ (assuming adjacency list)
- So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

Doing Better

Avoid searching for a zero-degree node every time!

- Keep the “pending” zero-degree nodes in a list, stack, queue, bag, or something that gives O(1) add/remove
- Order we process them affects the output but not correctness or efficiency

Using a queue:

- Label each vertex with its in-degree,
- Enqueue all 0-degree nodes
- While queue is not empty
  - v = dequeue()
  - Output v and remove it from the graph
  - For each vertex u adjacent to v, decrement the in-degree of u and if new degree is 0, enqueue it

Running Time?

```
labelAllWithIndegreesAndEnqueueZeros();

for(i=0; i < numVertices; i++) {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
       w.indegree--;
       if(w.indegree==0)
          enqueue(w);
   }
}
```

- Initialization: $O(|V| + |E|)$ (assuming adjacency list)
- Sum of all enqueues and dequeues: $O(|V|)$
- Sum of all decrements: $O(|E|)$ (assuming adjacency list)
- So total is $O(|E| + |V|)$ – much better for sparse graph!

More Graph Algorithms

Finding a shortest path is one thing

- What happens when we consider weighted edges (as in distances)?

Next time we will discuss shortest path algorithms and contributions of a curmudgeonly computer scientist