CSE 332 Data Abstractions: Graphs and Graph Traversals

Kate Deibel

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Last Time

We introduced the idea of graphs and their associated terminology

Key terms included:

- Directed versus Undirected
- Weighted versus Unweighted
- Cyclic or Acyclic
- Connected or Disconnected
- Dense or Sparse
- Self-loops or not

These are all important concepts to consider when implementing a graph data structure

Graph Data Structures

The two most common graph data structures

- Adjacency Matrix
- Adjacency List

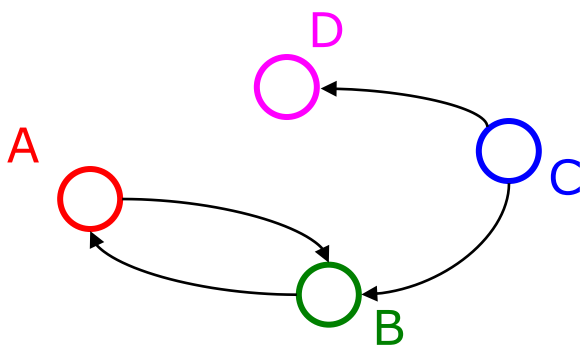
Whichever is best depends on the type of graph, its properties, and what you want to do with the graph

Adjacency Matrix

Assign each node a number from 0 to |V|-1

A |V| x |V| matrix of Booleans (or 0 versus 1)

- Then M[u][v]==true → an edge exists from u to v
- This example is for a directed graph



From \ To	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

Run time to get a vertex v's out-edges?

    O(|V|) → iterate over v's row

Run time to get a vertex v's in-edges?

    O(|V|) → iterate over v's column

Run time to decide if an edge (u,v) exists?

    O(1) → direct lookup of M[u][v]

Run time to insert an edge (u,v)?

    O(1) → set M[u][v] = true

Run time to delete an edge (u,v)?

    O(1) → set M[u][v] = false

Space requirements:

    O(|V|²) → 2-dimensional array

Best for sparse or dense graphs?

    Dense → We have to store every possible edge!!

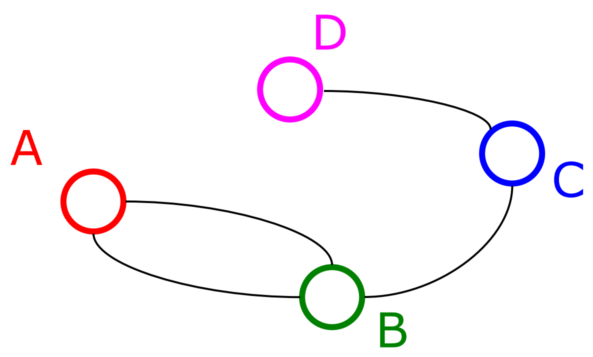
To

From	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix: Undirected Graphs

How will the adjacency matrix work for an undirected graph?

- Will be symmetric about diagonal axis
- Save space by using only about half the array?



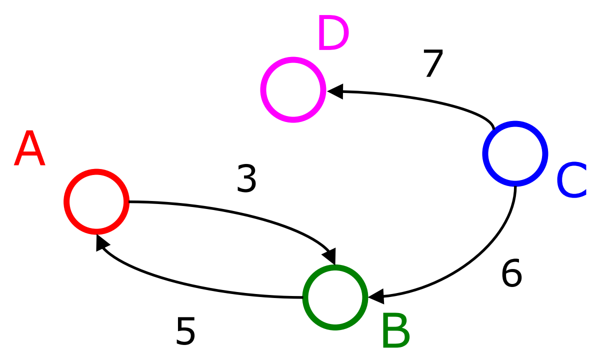
	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	*F*	*T*	*F*	T
D	*F*	*F*	*T*	*F*

- But how would you "get all neighbors"?

Adjacency Matrix: Weighted Graphs

How will the adjacency matrix work for a weighted graph?

- Instead of Boolean, store a number in each cell
- Need some value to represent 'not an edge'
  - 0, -1, or some other value based on how you are using the graph
  - Might need to be a separate field if no restrictions on weights

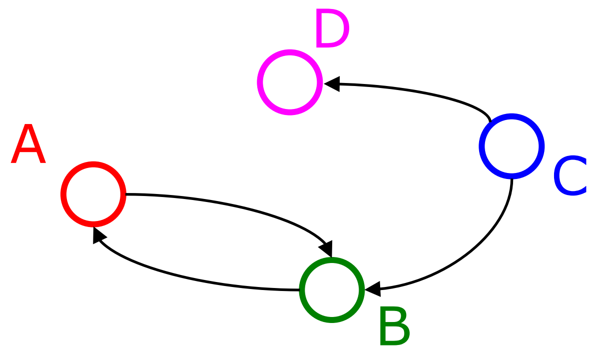


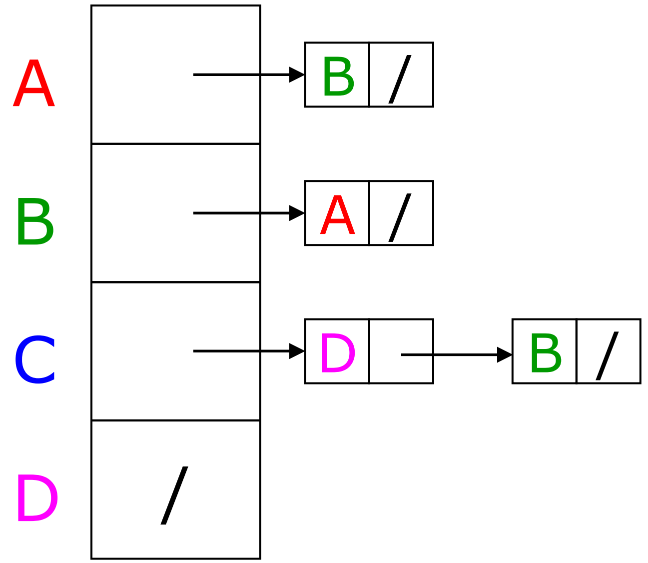
	A	B	C	D
A	0	3	0	0
B	5	0	0	0
C	*0*	*6*	*0*	7
D	*0*	*0*	*0*	*0*

Adjacency List

Assign each node a number from `0` to `|V|-1`

- An array of length `|V|` in which each entry stores a list of all adjacent vertices (e.g., linked list)
- This example is again for a directed graph





Adjacency List Properties

Run time to get a vertex v's out-edges?

O(d) → where *d* is v's out-degree

Run time to get a vertex v's in-edges?

O(|E|) → check every vertex list (or keep a second list for in-edges)

Run time to decide if an edge (u,v) exists?

O(d) → where *d* is u's out-degree

Run time to insert an edge (u,v)?

O(1) → unless you need to check if it's already there

Run time to delete an edge (u,v)?

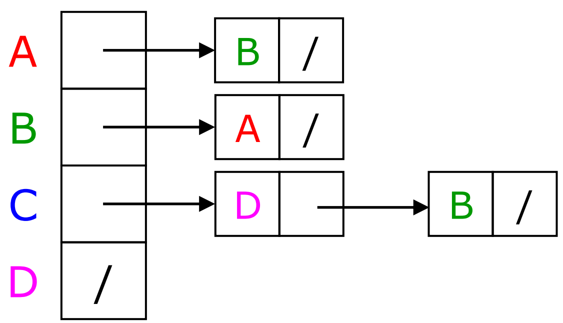
O(d) → where *d* is u's out-degree

Space requirements:

O(|V|+|E|) → vertex array plus edge nodes

Best for sparse or dense graphs?

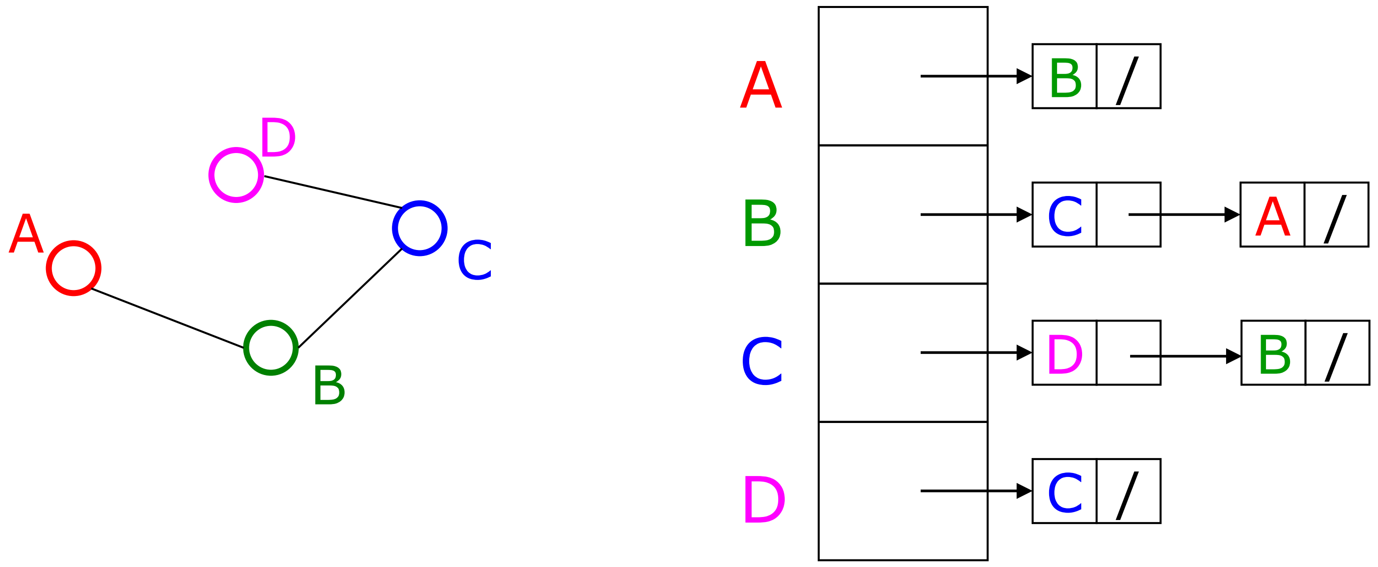
Sparse → Only store the edges needed



Adjacency List: Undirected Graphs

Adjacency lists also work well for undirected graphs with one caveat

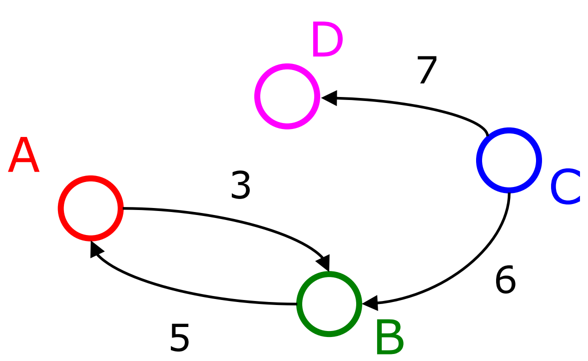
- Put each edge in two lists to support efficient "get all neighbors"
- Only an additional O(|E|) space

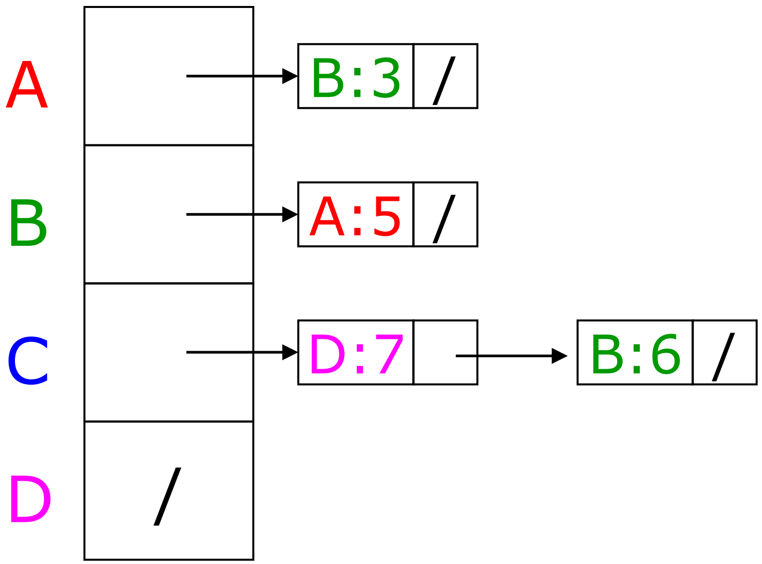


Adjacency List: Weighted Graphs

Adjacency lists also work well for weighted graphs but where do you store the weights?

- In a matrix? → $O(|V|^2)$ space
- Store a weight at each node in list → $O(|E|)$ space





Which is better?

Graphs are often sparse

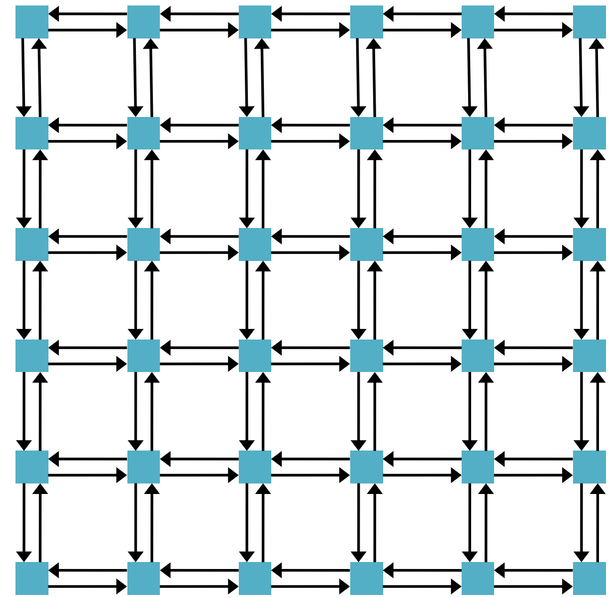
- Streets form grids
- Airlines rarely fly to all cities

Adjacency lists generally the better choice

- Slower performance
- **HUGE** space savings

How Huge of Space Savings?

Consider this 6x6 city street grid:



$|V| = 36$

$|E| = 6 \times 5 \times 2 + 6 \times 5 \times 2 = 120$

Adjacency Matrix: $O(|V|^2)$
→ $36^2 = 1296$

Adjacency List: $O(|E| + |V|)$
→ $36 + 2 \times 120 = 276$ (we'll store both in and out-edges)

Savings Factor = 276/1296 = 23/108 ≈ 21% of the space

In general, savings are:

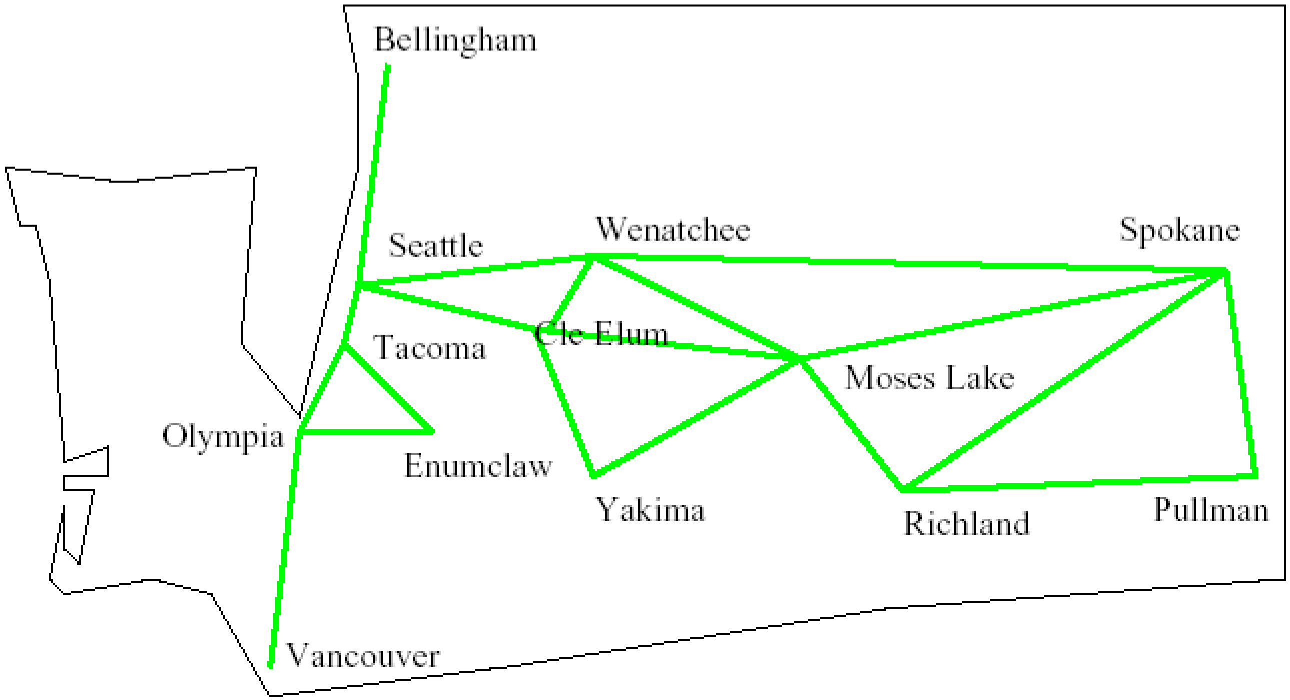
$$\frac{V+E}{V^2} = \frac{1}{V} + \frac{E}{V^2}$$

Recall that a sparse graph has $|E|=o(|V|^2)$, strictly less than quadratic

Might be easier to list what isn't a graph application…

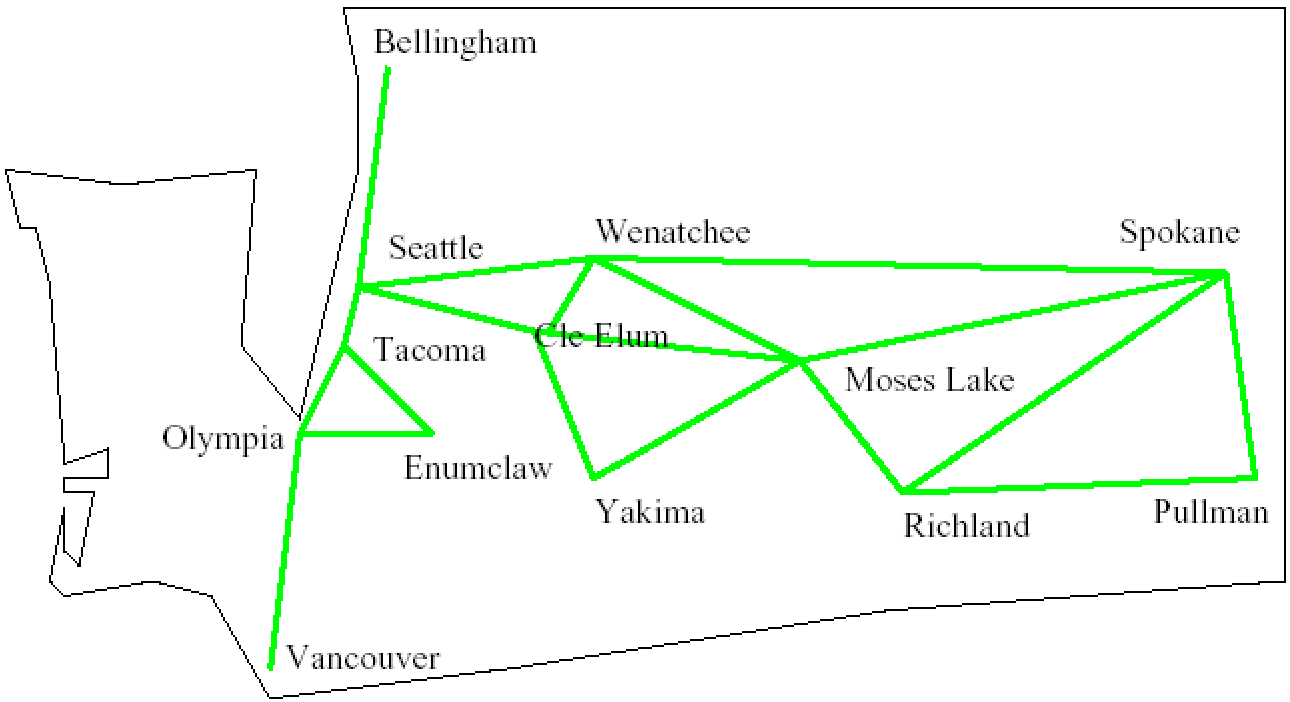
GRAPH APPLICATIONS: TRAVERSALS

Application: Moving Around WA State



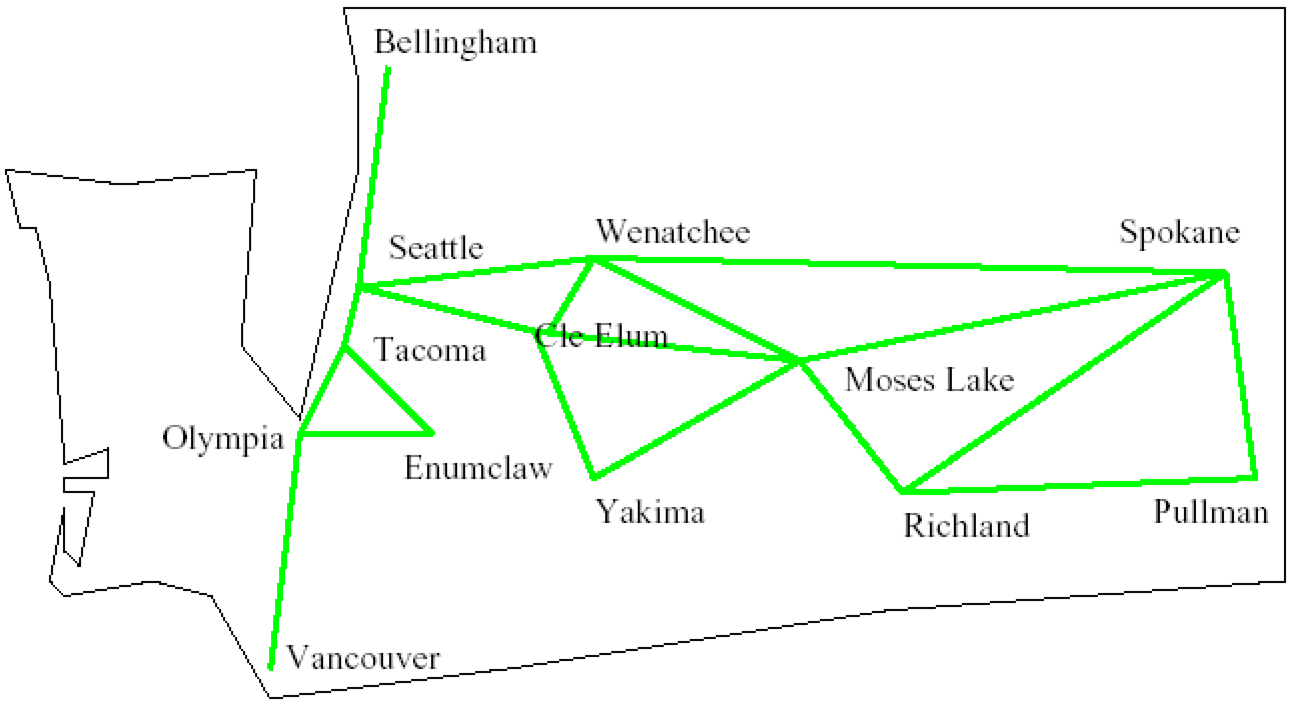
What's the *shortest way* to get from Seattle to Pullman?

Application: Moving Around WA State



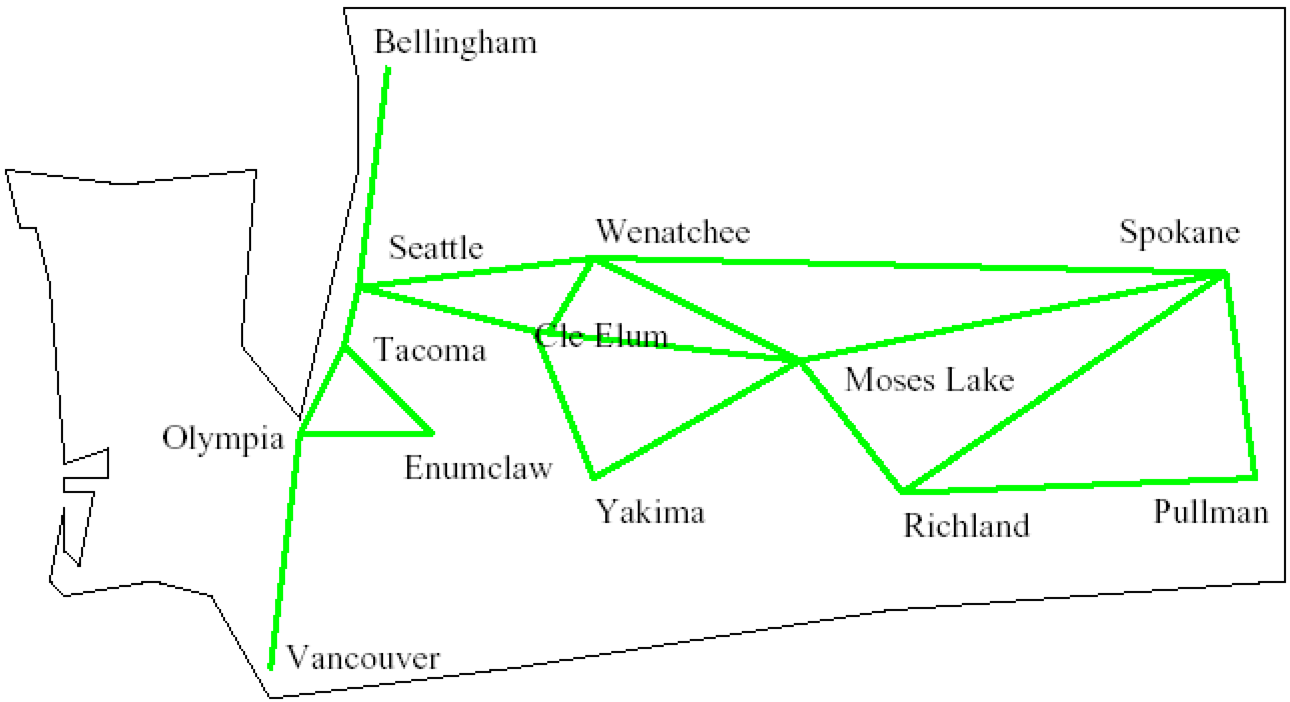
What's the *fastest way* to get from Seattle to Pullman?

Application: Communication Reliability



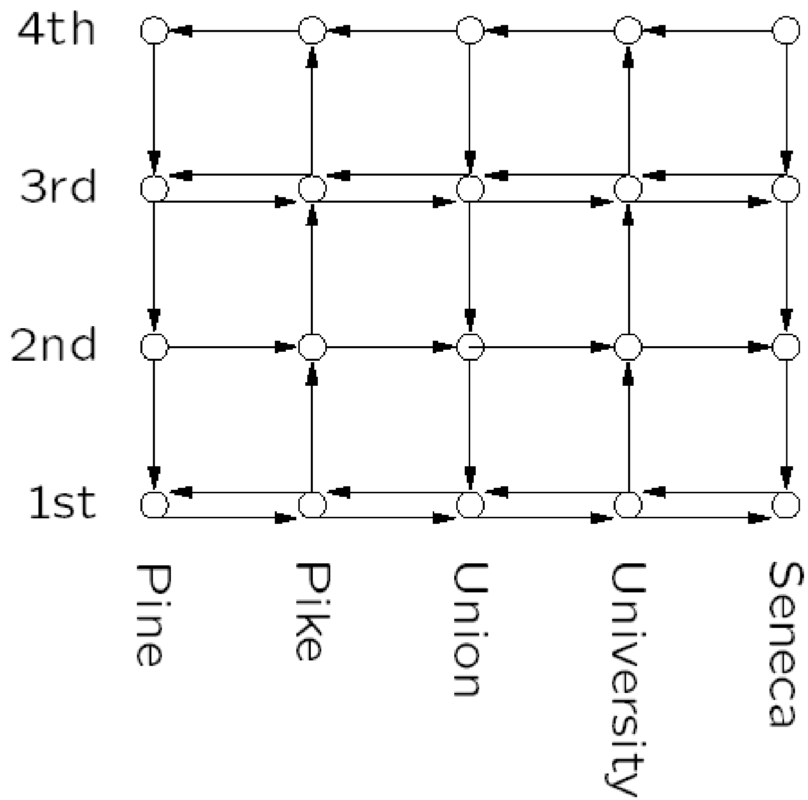
If Wenatchee's phone exchange *goes down*, can Seattle still talk to Pullman?

Application: Communication Reliability



If Tacoma's phone exchange *goes down*, can Olympia still talk to Spokane?

Applications: Bus Routes Downtown



If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

Graph Traversals

For an arbitrary graph and a starting node v, find all nodes reachable from v (i.e., there exists a path)

- Possibly "do something" for each node (print to output, set some field, return from iterator, etc.)

Related Problems:

- Is an undirected graph connected?
- Is a digraph weakly/strongly connected?
  - For strongly, need a cycle back to starting node

Graph Traversals

Basic Algorithm for Traversals:

- Select a starting node
- Make a set of nodes adjacent to current node
- Visit each node in the set but "mark" each nodes after visiting them so you don't revisit them (and eventually stop)
- Repeat above but skip "marked nodes"

In Rough Code Form

```
traverseGraph(Node start) {
    Set pending = emptySet();
    pending.add(start)
    mark start as visited
    while(pending is not empty) {
        next = pending.remove()
        for each node u adjacent to next
            if(u is not marked) {
                mark u
                pending.add(u)
            }
        }
    }
}
```

Running Time and Options

BFS and DFS traversal are both O(|V|+|E|) if using and adjacency list

- Queue/stack insert removes are generally O(1)
- Adjacency lists make it O(|V|) to find neighboring vertices/edges
- We will mark every node → O(|V|)
- We will touch every edge at most twice → O(|E|)

Because |E| is generally at least linear to |V|, we usually just say BFS/DFS are O(|E|)

- Recall that in a connected graph |E|≥|V|-1

The Order Matters

The order we traverse depends entirely on how add and remove work/are implemented

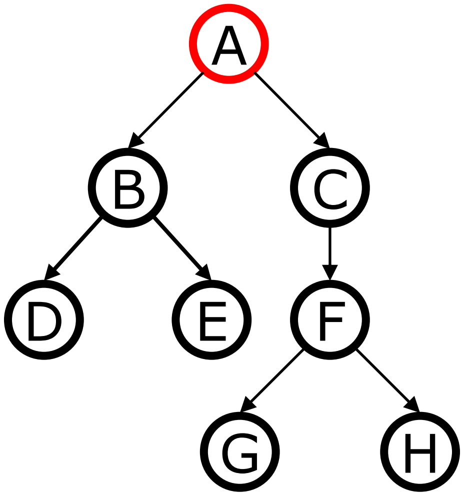
- DFS: a stack "depth-first graph search"
- BFS: a queue "breadth-first graph search"

DFS and BFS are "big ideas" in computer science

- Depth: recursively explore one part before going back to the other parts not yet explored
- Breadth: Explore areas closer to start node first

Recursive DFS, Example with Tree

A tree is a graph and DFS and BFS are particularly easy to "see" in one

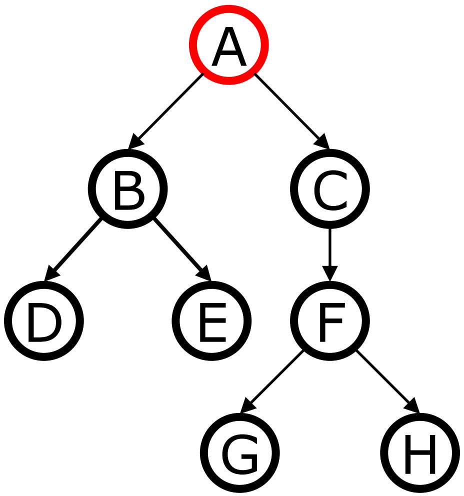



```
DFS(Node start) {
    mark and process start
    for each node u adjacent to start
        if u is not marked
            DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

- This is a "pre-order traversal" for trees
- The marking is unneeded here but because we support arbitrary graphs, we need a means to process each node exactly once

DFS with Stack, Example with Tree

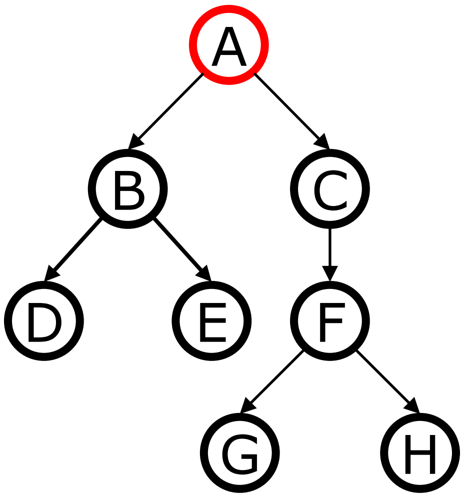



```
DFS2(Node start) {
   initialize stack s to hold start
   mark start as visited
   while(s is not empty) {
       next = s.pop() // and "process"
       for each node u adjacent to next
          if(u is not marked)
             mark u and push onto s
   }
}
```

Order processed: A, C, F, H, G, B, E, D

- A different order but still a perfectly fine traversal of the graph

BFS with Queue, Example with Tree




```
BFS(Node start) {
   initialize queue q to hold start
   mark start as visited
   while(q is not empty) {
      next = q.dequeue() // and "process"
      for each node u adjacent to next
         if(u is not marked)
            mark u and enqueue onto q
   }
}
```

Order processed: A, B, C, D, E, F, G, H

- A "level-order" traversal

DFS/BFS Comparison

BFS always finds the shortest path/optimal solution from the start vertex to the target

- Storage for BFS can be extremely large
- A k-nary tree of height h could result in a queue size of k^h

DFS can use less space in finding a path

- If longest path in the graph is p and highest out-degree is d then DFS stack never has more than $d \cdot p$ elements

Implications

For large graphs, DFS is more memory efficient, *if we can limit the maximum path length to some fixed d.*

If we *knew* the distance from the start to the goal in advance, we could simply *not add any children to stack after level d*

But what if we don't know *d* in advance?

Iterative Deepening (IDFS)

Algorithms

- Try DFS up to recursion of K levels deep.
- If fail, increment K and start the entire search over

Performance:

- Like BFS, IDFS finds shortest paths
- Like DFS, IDFS uses less space
- Some work is repeated but minor compared to space savings

Saving the Path

Our graph traversals can answer the standard *reachability* question:

"Is there a path from node x to node y?"

But what if we want to actually output the path?

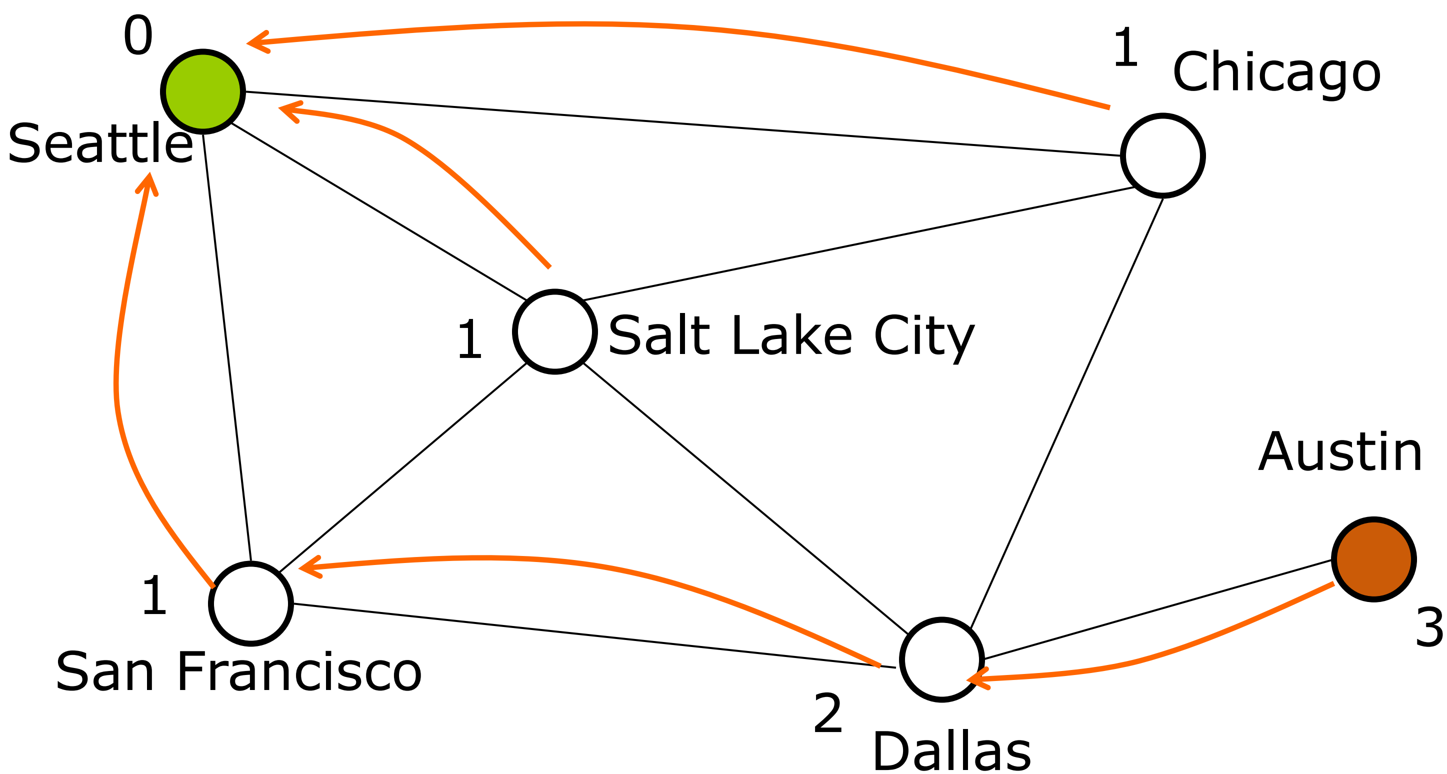
Easy:

- Store the previous node along the path: When processing *u* causes us to add *v* to the search, set *v.path* field to be *u*)
- When you reach the goal, follow path fields back to where you started (and then reverse the answer)
- What's an easy way to do the reversal? A Stack!!

Example using BFS

What is a path from Seattle to Austin?

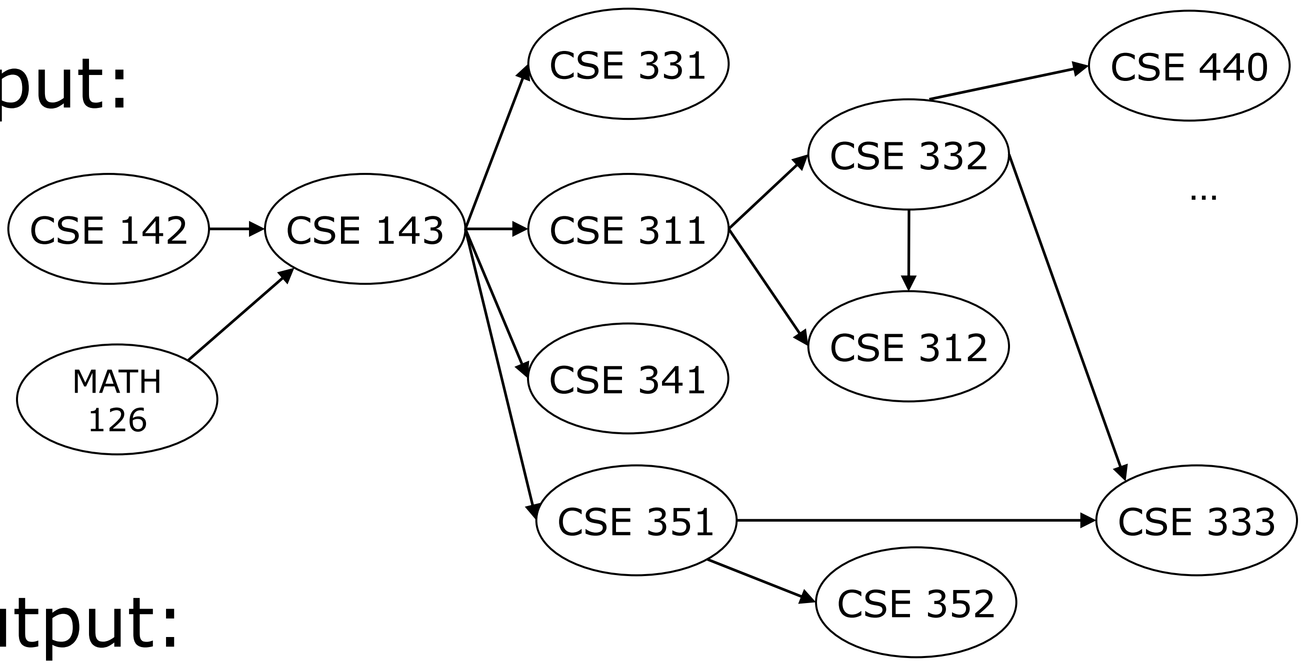
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



Topological Sort

Problem: Given a DAG $G=(V, E)$, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:

- 142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352

Disclaimer: Do not use for official advising purposes!
(Implies that CSE 332 is a pre-req for CSE 312 – not true)

Questions and Comments

Terminology:

A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Why do we perform topological sorts only on DAGs?

- Because a cycle means there is no correct answer

Is there always a unique answer?

- No, there can be one or more answers depending on the provided graph

What DAGs have exactly 1 answer?

- Lists

Uses Topological Sort

Figuring out how to finish your degree

Computing the order in which to recalculate cells in a spreadsheet

Determining the order to compile files with dependencies

In general, use a dependency graph to find an allowed order of execution

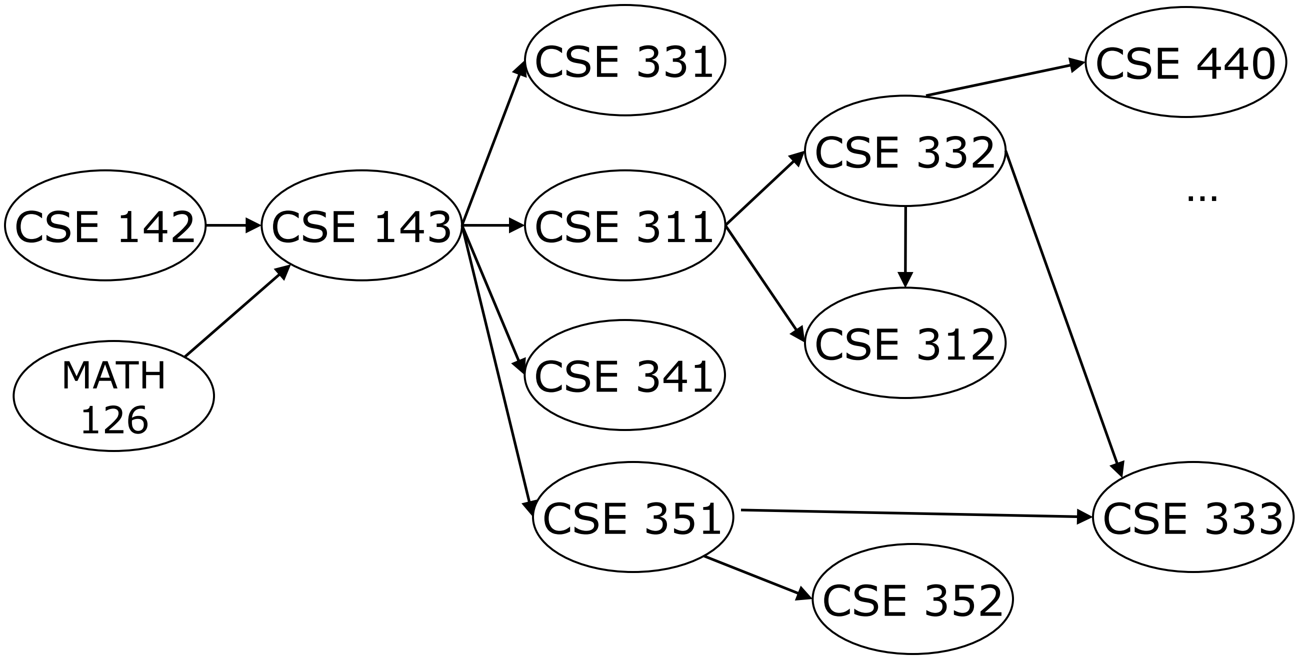
Topological Sort: First Approach

1. Label each vertex with its in-degree
   - Think "write in a field in the vertex"
   - You could also do this with a data structure on the side

2. While there are vertices not yet outputted:
   a) Choose a vertex **v** labeled with in-degree of 0
   b) Output **v** and "remove it" from the graph
   c) For each vertex **u** adjacent to **v**, decrement in-degree of **u**
      - (i.e., **u** such that (**v**,**u**) is in **E**)

Example

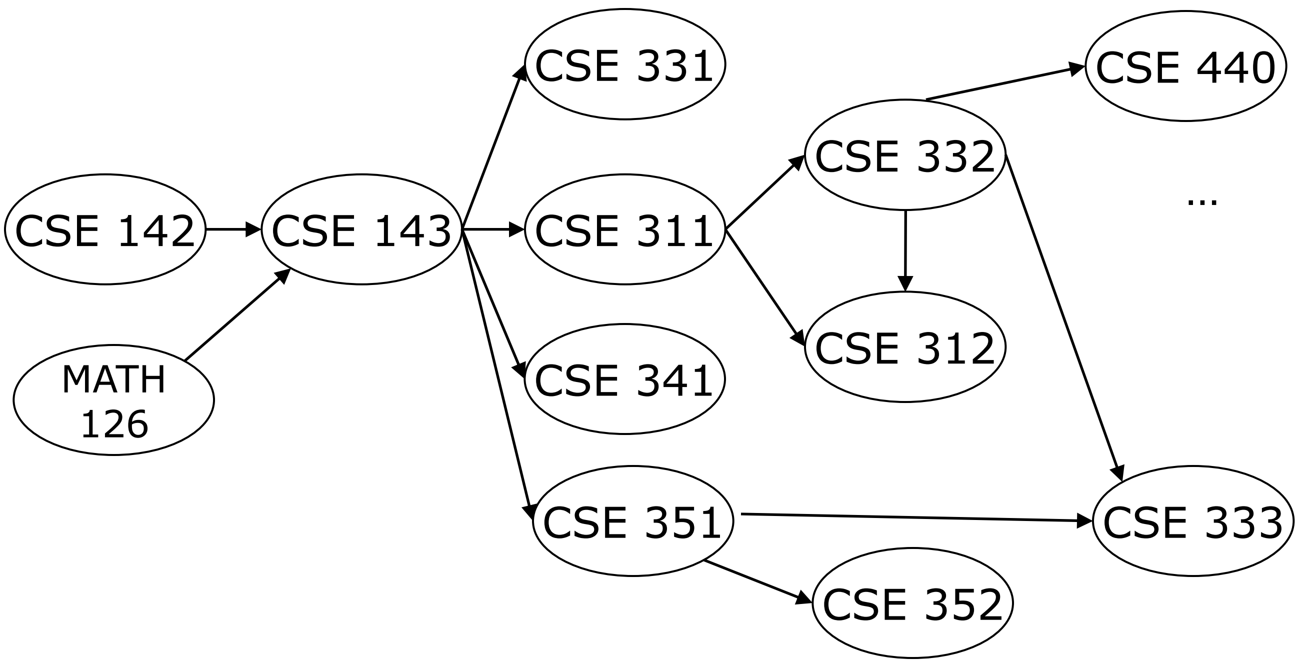
Output:




```
Node:     126 142 143 311 312 331 332 333 341 351 352 440
Removed?
In-deg:
```

Example

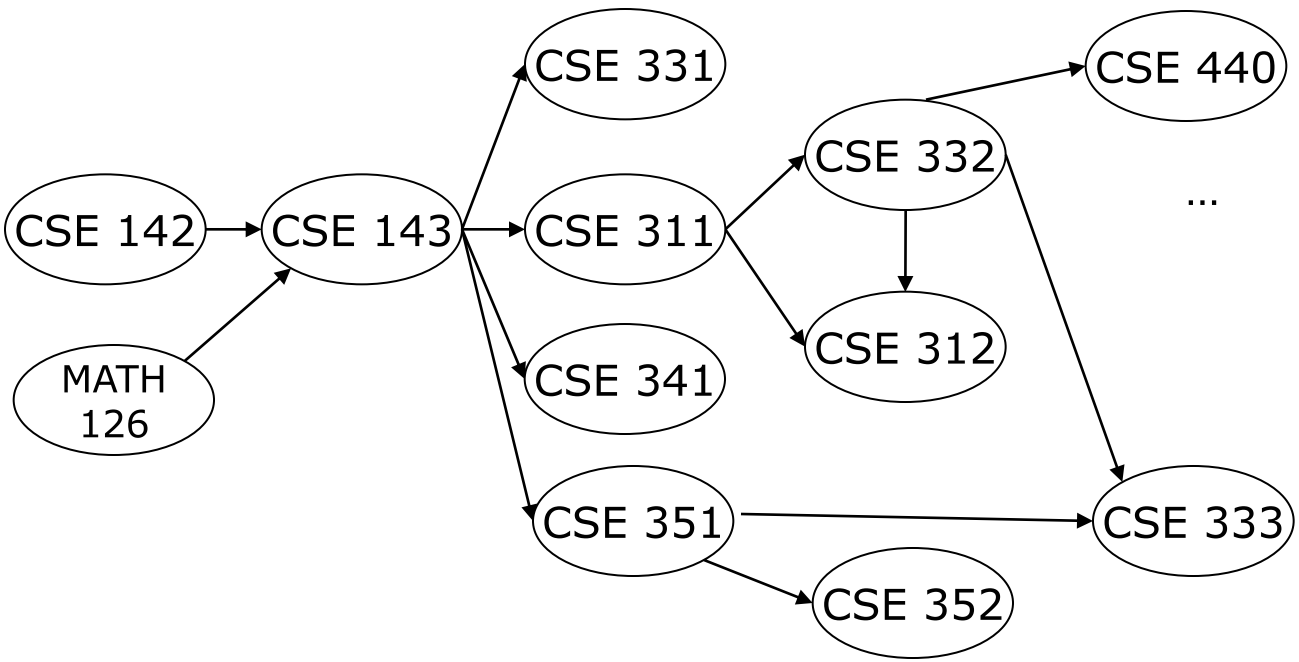
Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1

Example

Output:
126



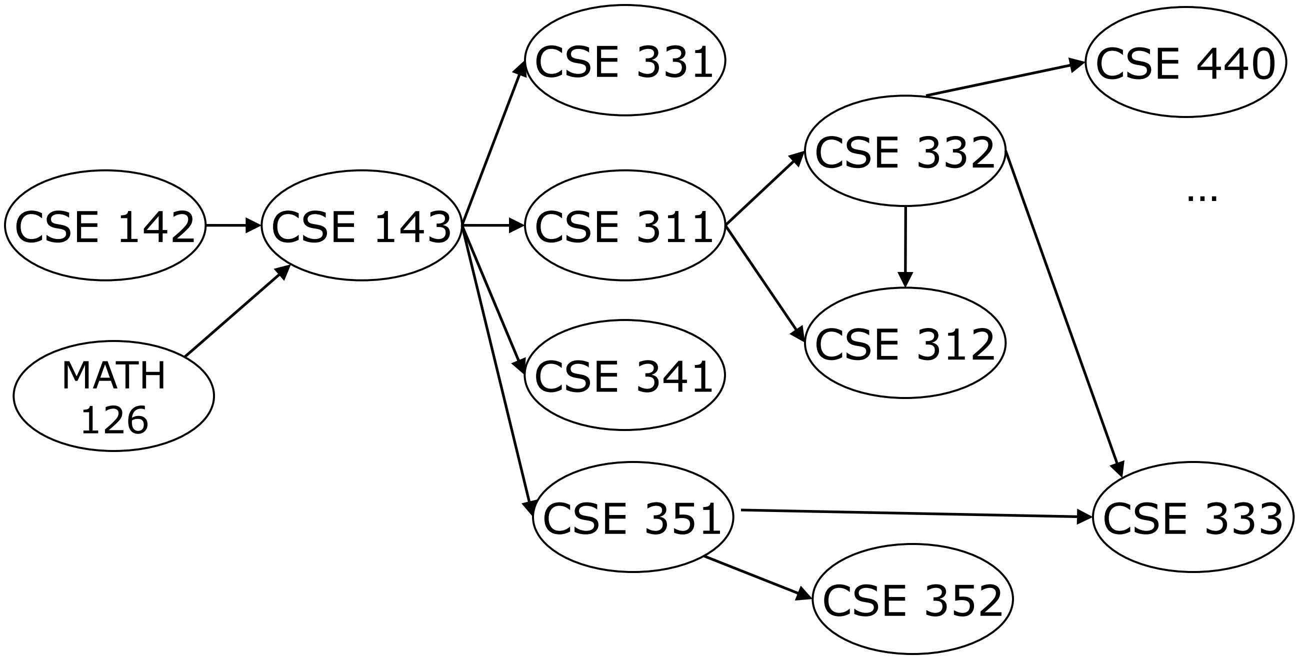
Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1									

Example

Output:

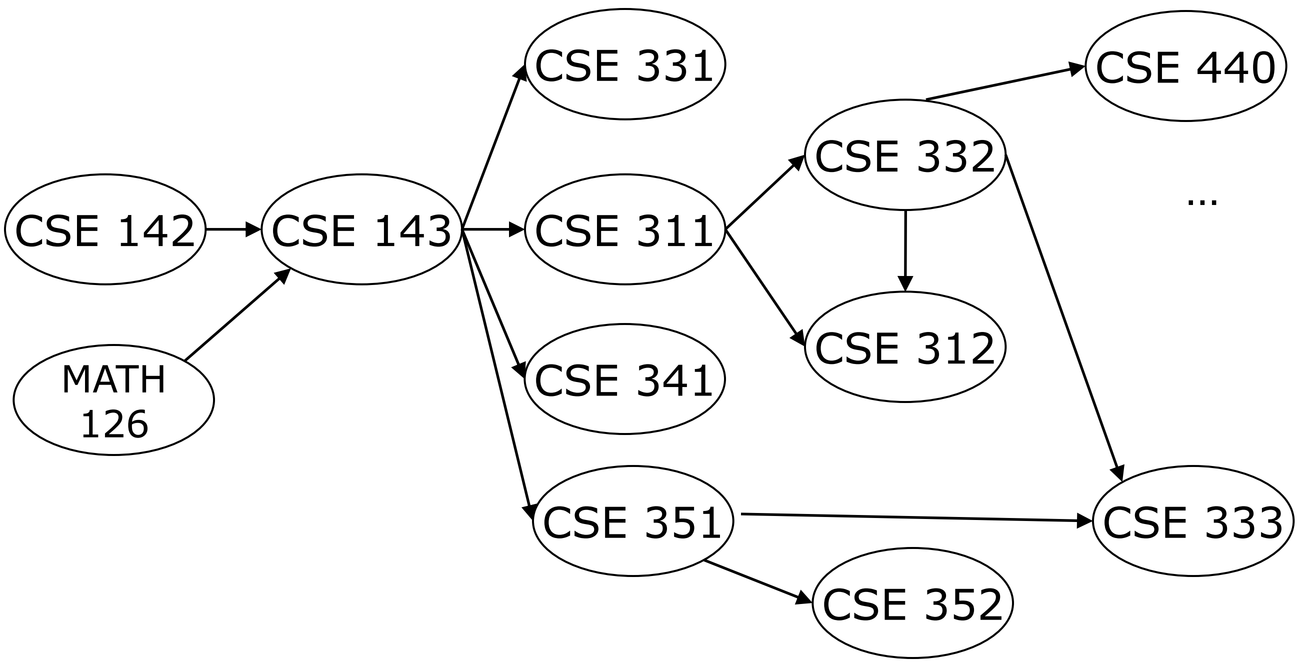
126

142



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

Example

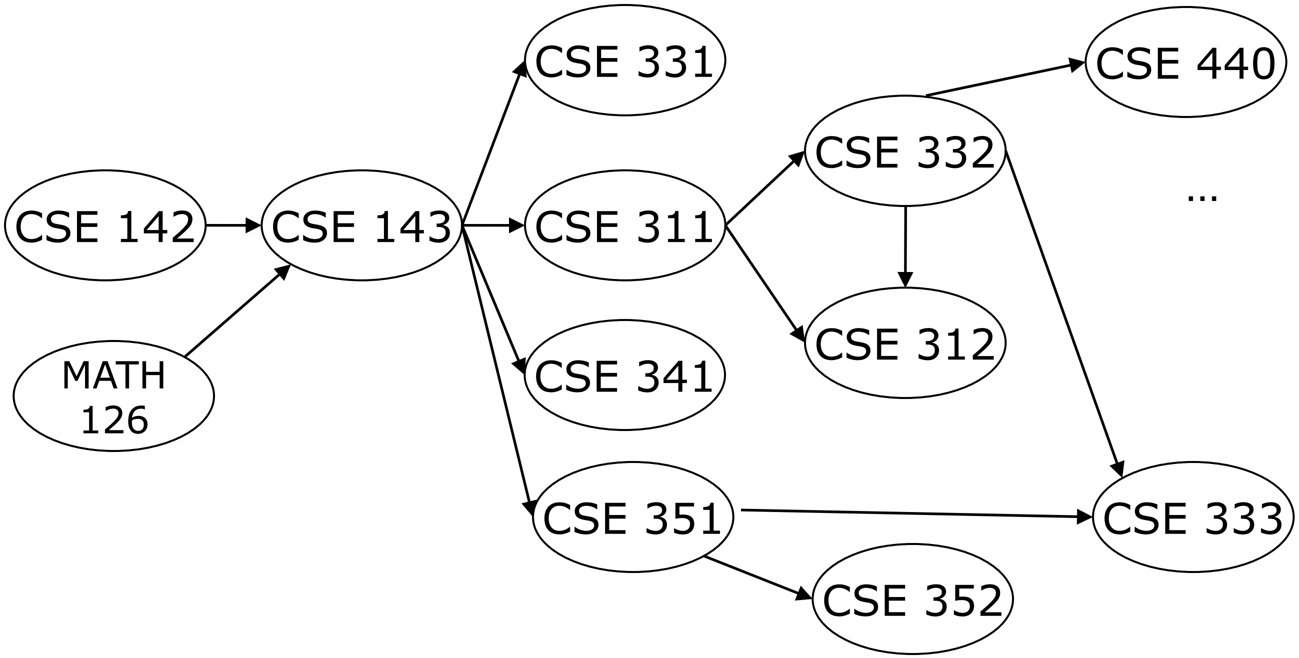


Output:
126
142
143

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x									
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

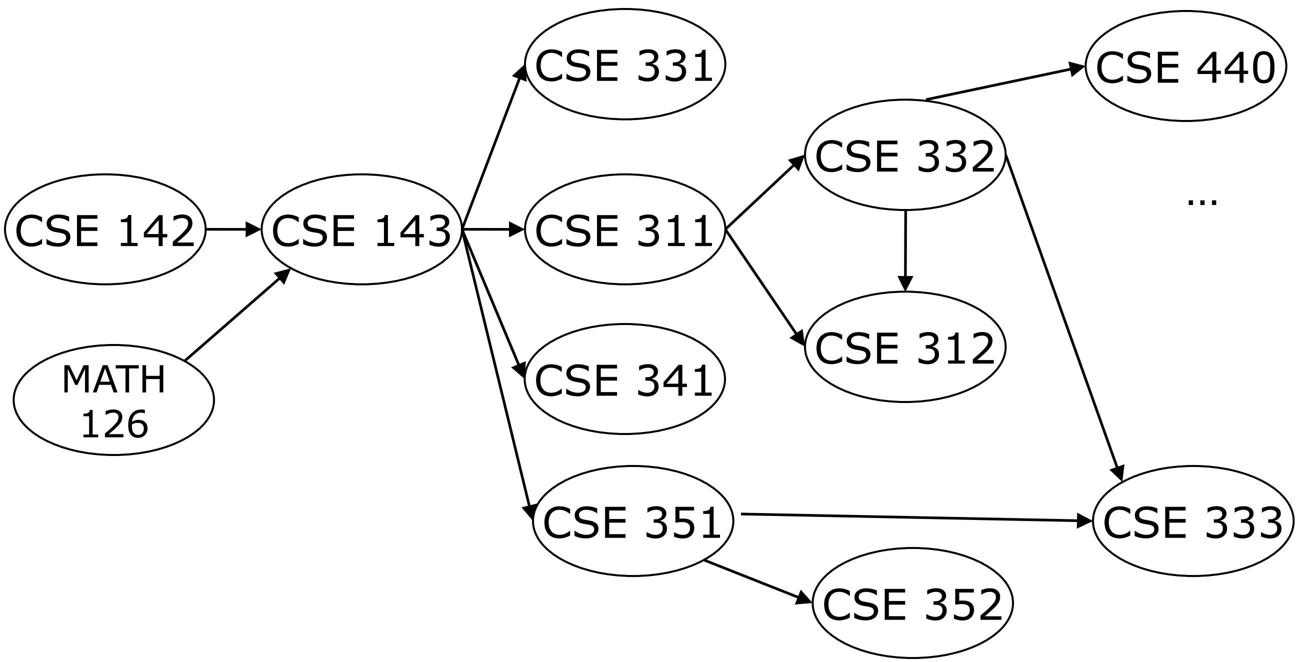
Example

Output:
126
142
143
311



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x								
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

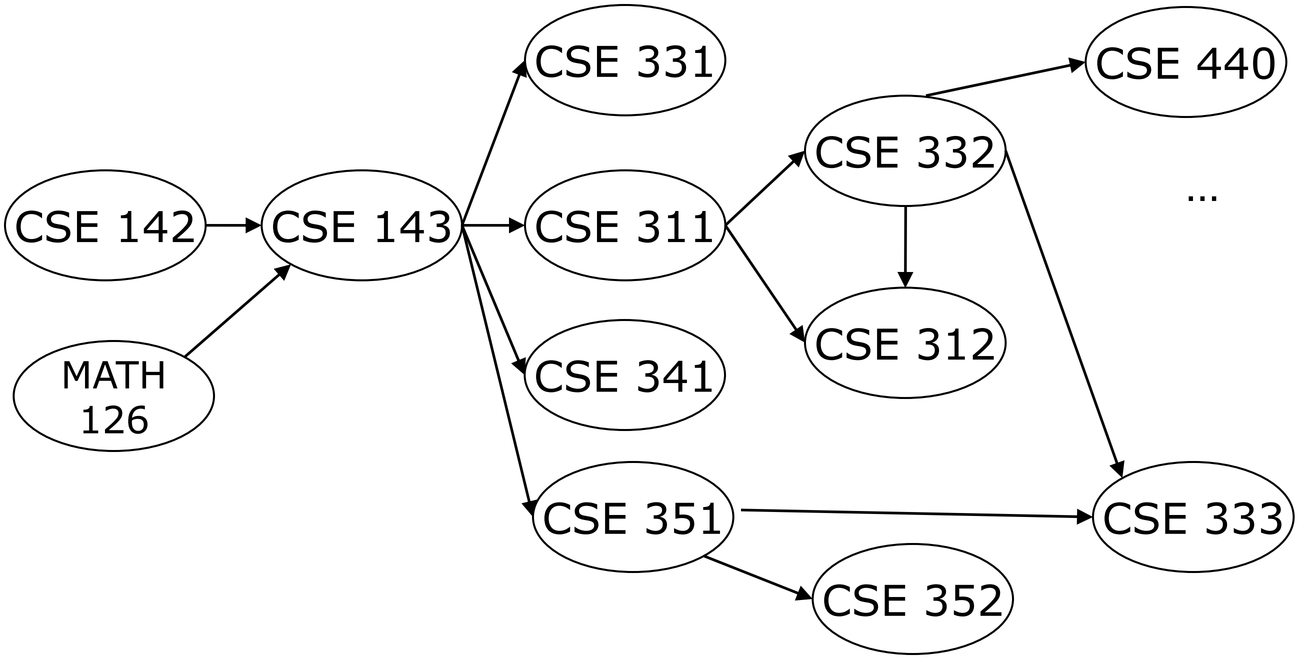
Example



Output:
126
142
143
311
331

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x						
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

Example



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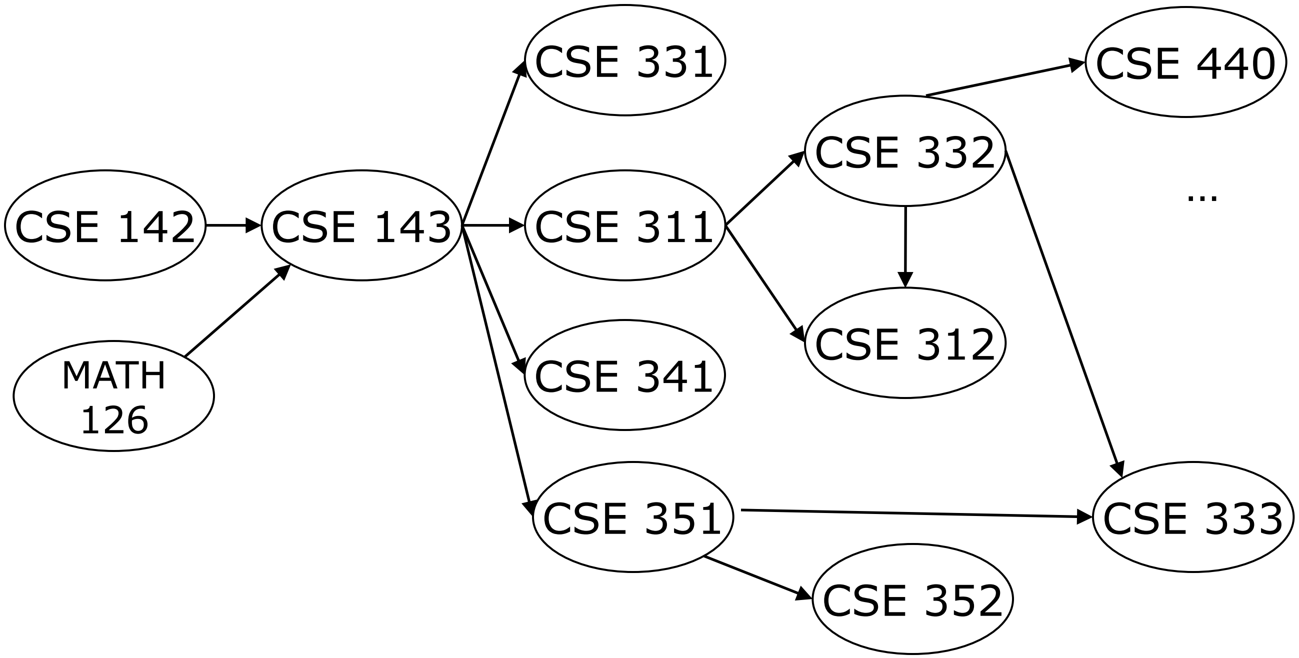
Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x	x					
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

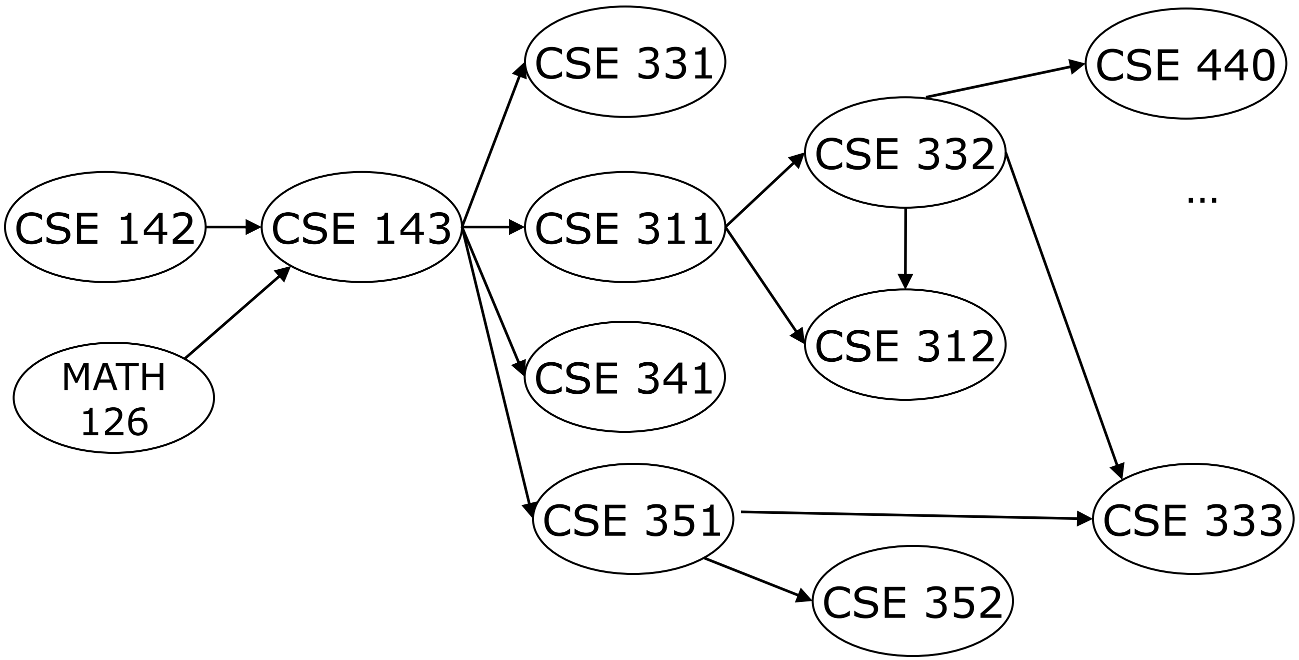
Example



Output:
126
142
143
311
331
332
312

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x					
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example



Output:
126
142
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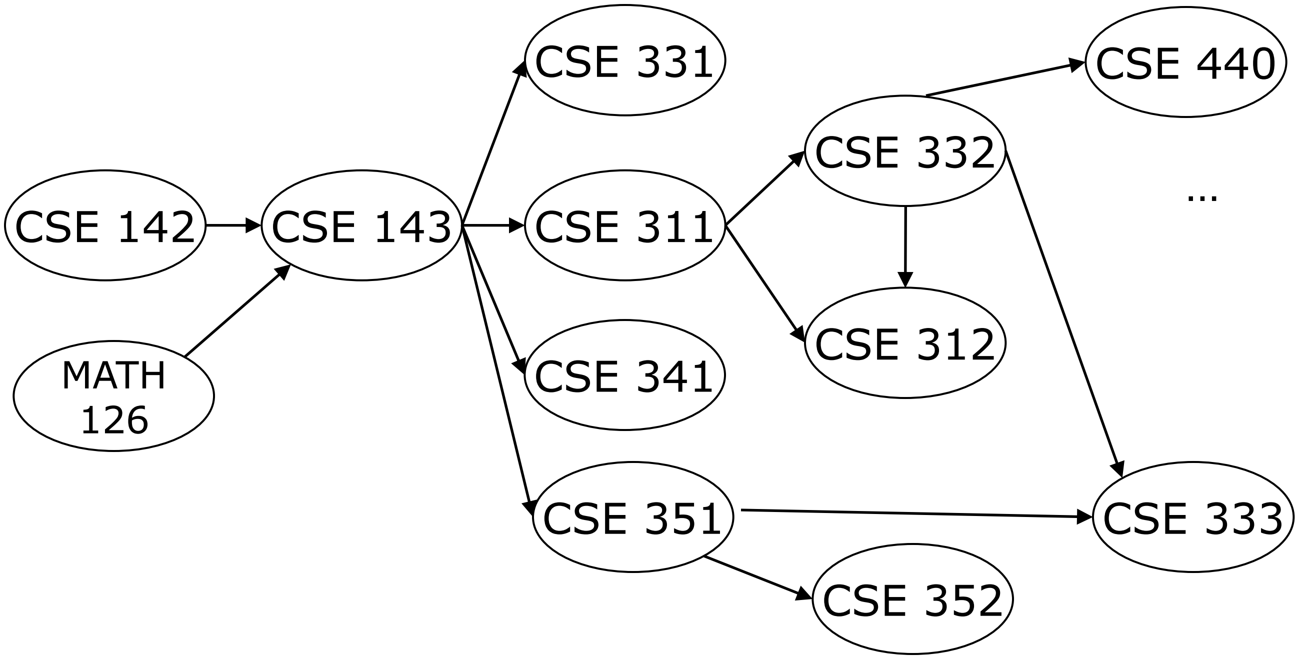
311
331
332
312
341

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x			
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example



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126
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331
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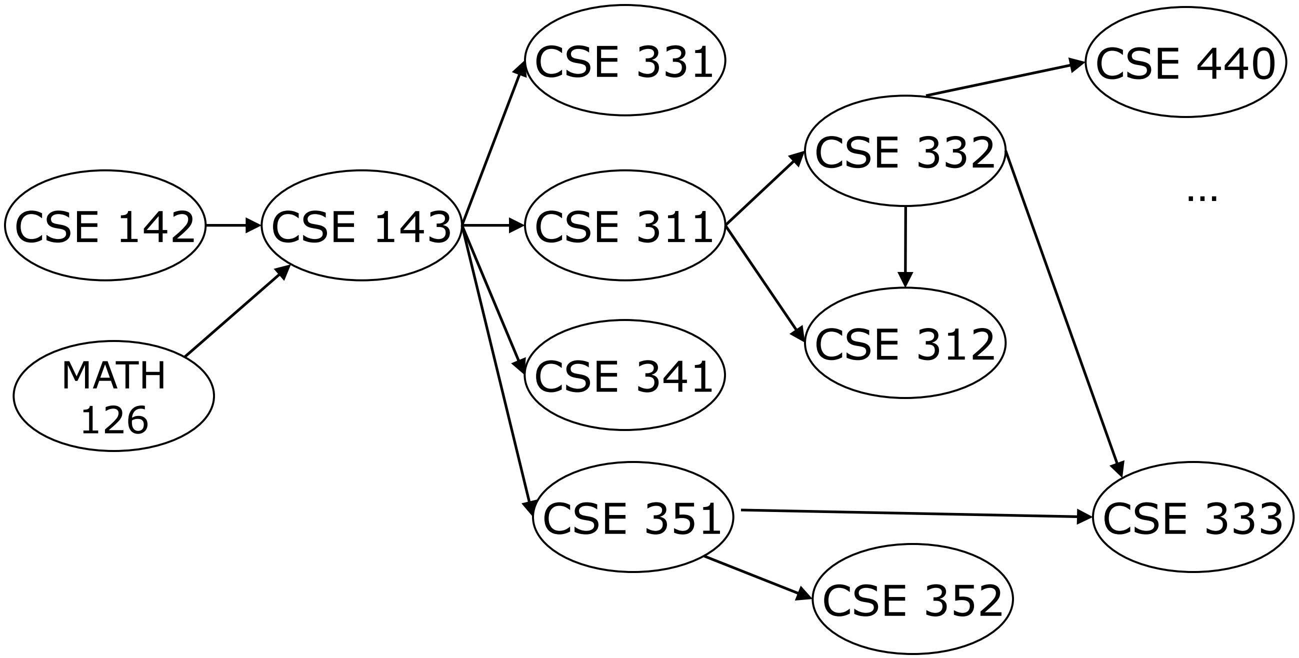
312
341
351

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x	x		
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



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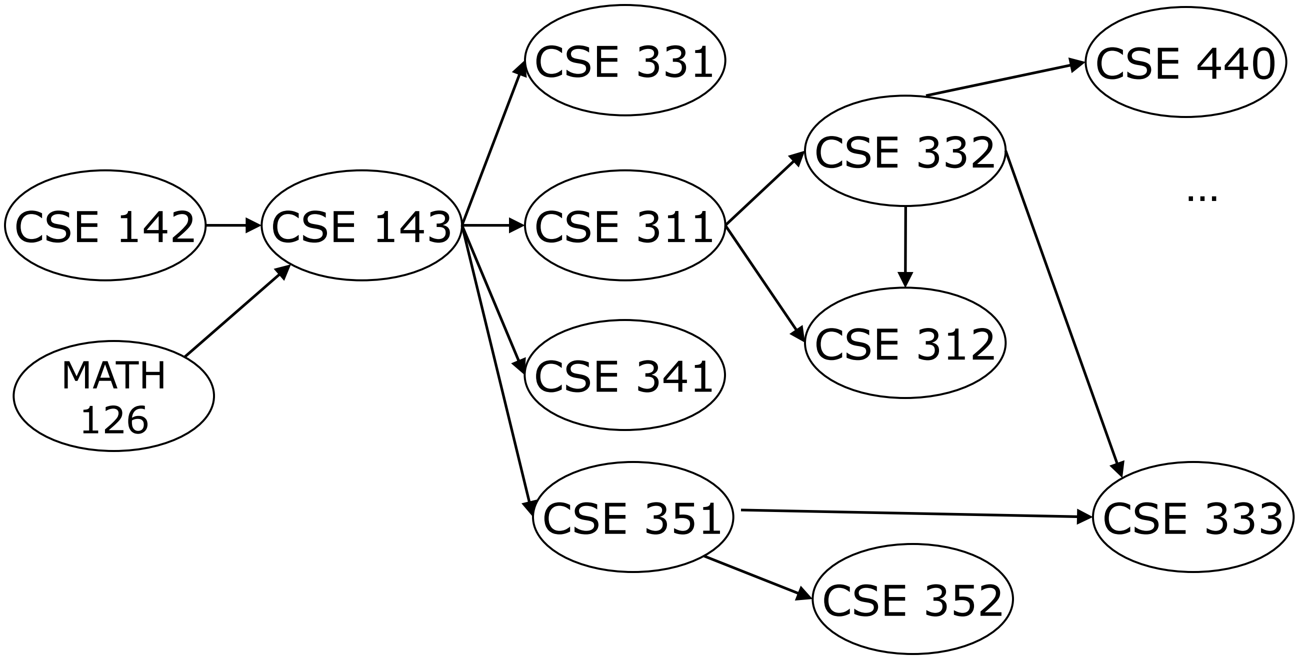
312
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x		
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example

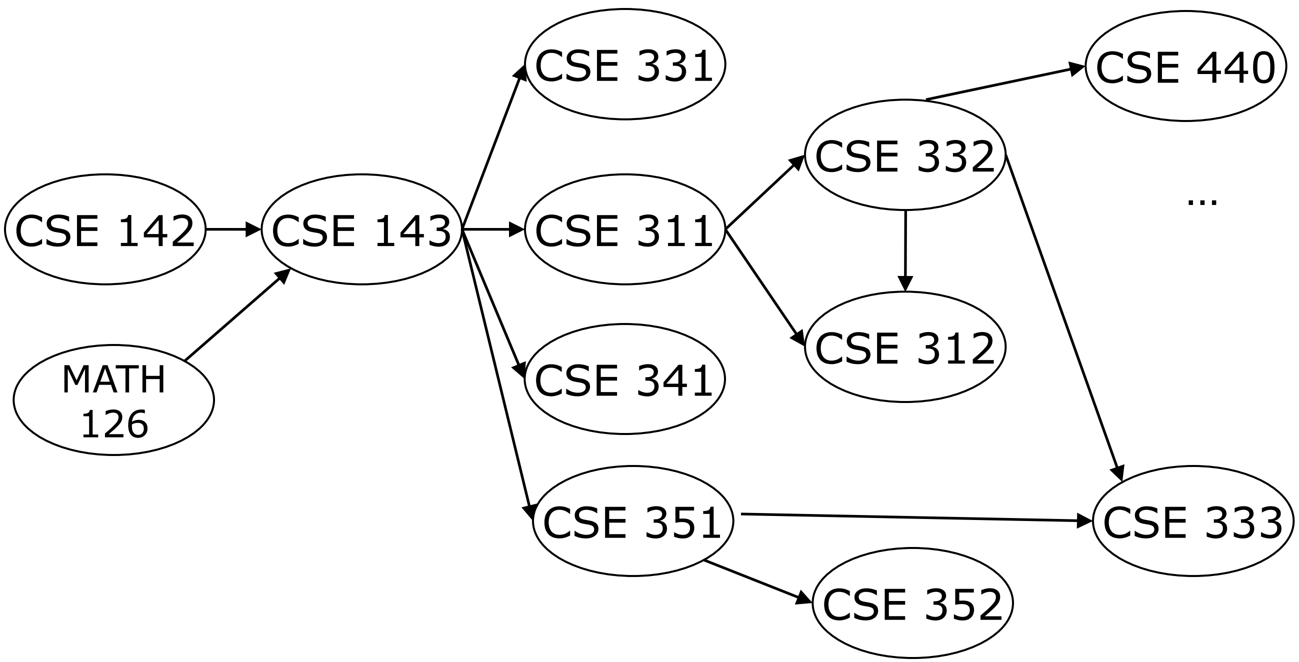


Output:

126 352
142
143
311
331
332
312
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



Output:

126 352
142 440
143
311
331
332
312
341
351
333

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-deg:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Running Time?

```
labelEachVertexWithItsInDegree();

for(i=0; i < numVertices; i++) {
      v = findNewVertexOfDegreeZero();
      put v next in output
      for each w adjacent to v
         w.indegree--;
}
```

What is the worst-case running time?

- Initialization $O(|V| + |E|)$ (assuming adjacency list)
- Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
- Sum of all decrements $O(|E|)$ (assuming adjacency list)
- So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

Doing Better

Avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, or something that gives O(1) add/remove
- Order we process them affects the output but not correctness or efficiency

Using a queue:

- Label each vertex with its in-degree,
- Enqueue all 0-degree nodes
- While queue is not empty
  - v = dequeue()
  - Output v and remove it from the graph
  - For each vertex u adjacent to v, decrement the in-degree of u and if new degree is 0, enqueue it

Running Time?

```
labelAllWithIndegreesAndEnqueueZeros();

for(i=0; i < numVertices; i++) {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
       w.indegree--;
       if(w.indegree==0)
          enqueue(w);
   }
}
```

- Initialization: $O(|V| + |E|)$ (assuming adjacency list)
- Sum of all enqueues and dequeues: $O(|V|)$
- Sum of all decrements: $O(|E|)$ (assuming adjacency list)
- So total is $O(|E| + |V|)$ – much better for sparse graph!

What about connectedness?

What happens if a graph is disconnected?

- With DFS?
- With BFS?
- With Topological Sorting?

All of these can be used to find connected components of the graph

- One just needs to start a new search at an unmarked node

Discovered by a most curmudgeonly man….

MOST COMMON TRAVERSAL: FINDING SHORTEST PATHS

Finding the Shortest Path

The graph traversals discussed so far work with path length (number of edges)but not path cost

Breadth-First Search found minimum path length from v to u in time O(|E|+(|V|)

- Actually, can find the minimum path length from v to every node
  - Still O(|E|+(|V|)
  - No faster way for a "distinguished" destination in the worst-case

Finding the Shortest Path

Question:

Given a graph G and two vertices v and u, what is the shortest path (shortest length) from v to u?

Solution:

Breadth-First Search starting at u will find minimum path length from v to u in time O(|E|+(|V|)

Actually, the search can be easily extended to find minimum path length from v to every node

- Still O(|E|+(|V|)
- No faster solution (in the worst-case) exists even if just focusing on one destination node

But That Was Path Length

Path length is the number of edges in a path

Path cost is sum of the weight of edges in a path

New Question:

Given a weighted graph and node v, what is the minimum-cost path from v to every node?

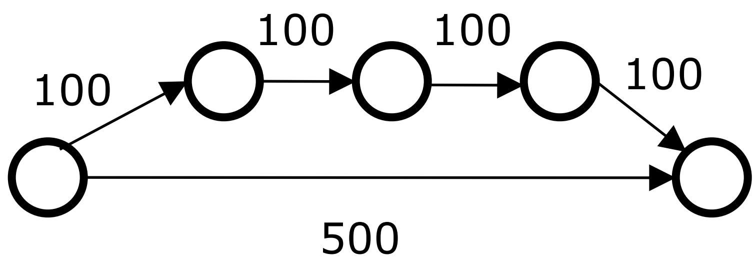
We could phrase this as from a node v to u, but it is asymptotically no harder than for one destination

Solution:

Let's try BFS… it worked before, right?

Why BFS Will Not Work

The shortest cost path may not have the fewest edges (shortest length)



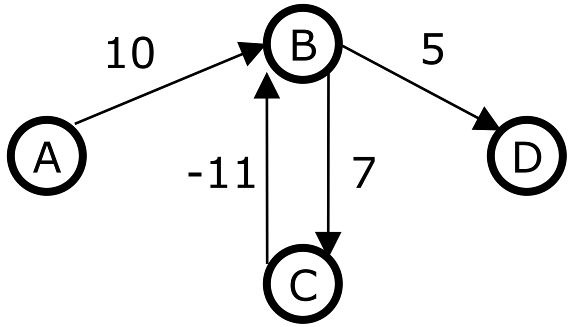
This happens frequently with airline tickets

- Which is why I travel through Atlanta all too often to get to Kentucky from Seattle

Regarding Negative Weights

Negative edge weights are a can of worms

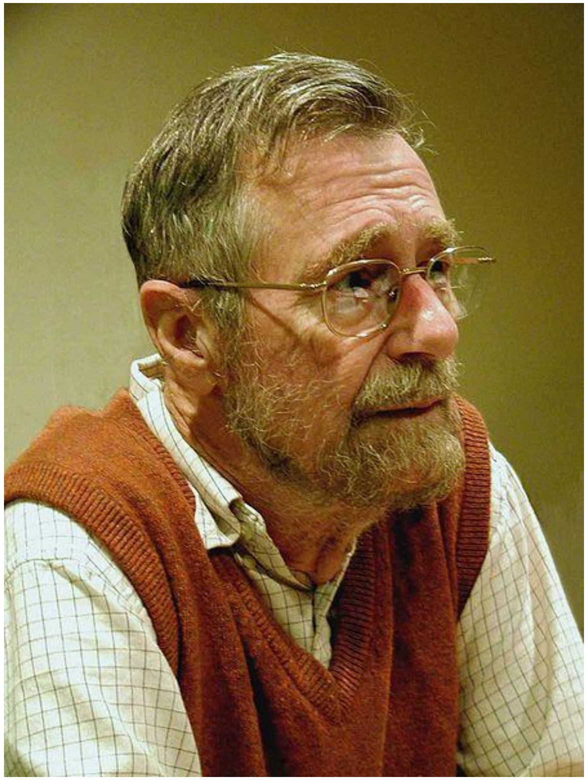
- If a cycle is negative, then the shortest path is -∞ (just repeat the cycle)



We will assume that there are no negative edge weights

- Today's algorithm gives erroneous results if edges can be negative

Dijkstra's Algorithm—The Man



Named after its inventor Edsger Dijkstra (1930-2002)

Truly one of the "founders" of computer science

This is just one of his many contributions

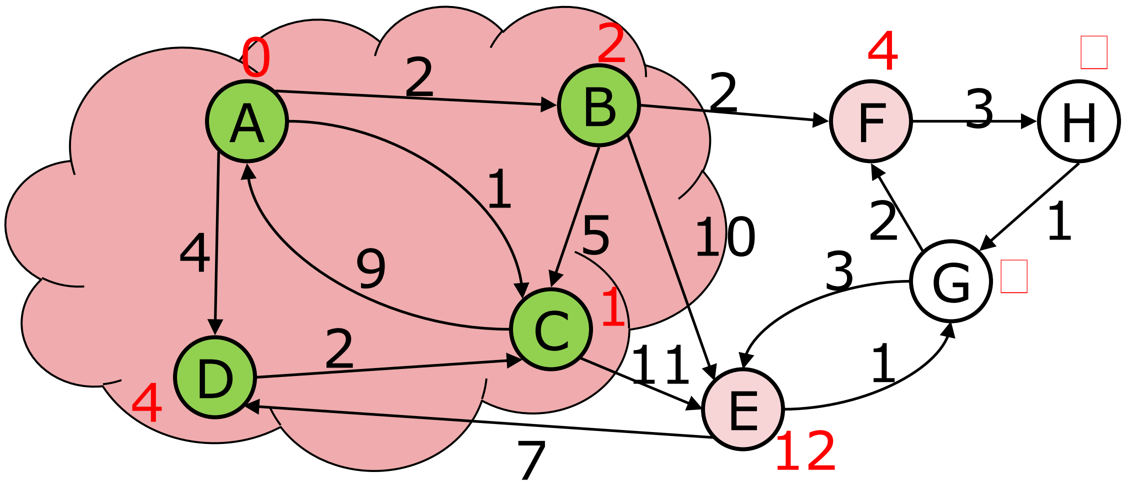
"Computer science is no more about computers than astronomy is about telescopes"

Dijkstra's Algorithm—The Idea

His algorithm is similar to BFS, but adapted to handle weights

- A priority queue will prove useful for efficiency
- Grow set of nodes whose shortest distance has been computed
- Nodes not in the set will have a "best distance so far"

Dijkstra's Algorithm—The Cloud



Initial State:

- Start node has cost 0
- All other nodes have cost ∞

At each step:

- Pick closest unknown vertex v
- Add it to the "cloud" of known vertices
- Update distances for nodes with edges from v

The Algorithm

1. For each node **v**≠**source**,
   Set **v**.*cost* = ∞ and **v**.*known* = false
2. Set **source**.*cost* = 0 and **source**.*known* = true
3. While there are unknown nodes in the graph
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known
   c) For each edge (**v**, **u**) with weight **w**,

      $\mathbf{c_1}$ = **v**.*cost* + **w** *// cost of best path through v to u*

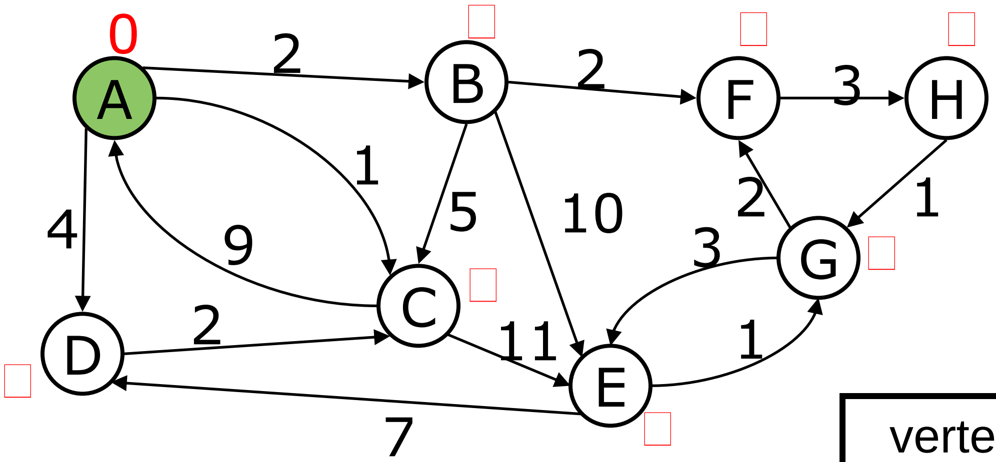
      $\mathbf{c_2}$ = **u**.*cost* *// cost of best path to u previously known*

      if($\mathbf{c_1} < \mathbf{c_2}$) *// if the path through v is better*

        u.*cost* = $\mathbf{c_1}$

        u.*path* = **v** *// for computing actual paths*

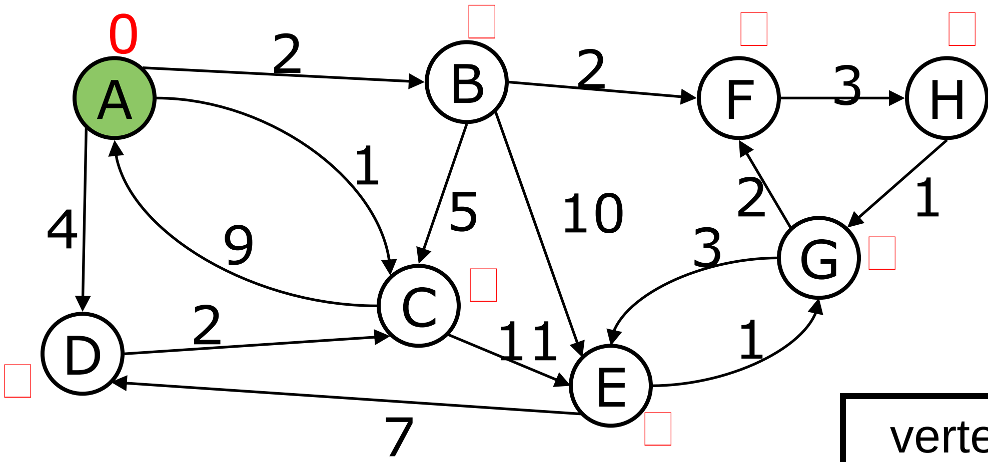
Example #1



Order Added to Known Set:

vertex	known?	cost	path
A			
B			
C			
D			
E			
F			
G			
H			

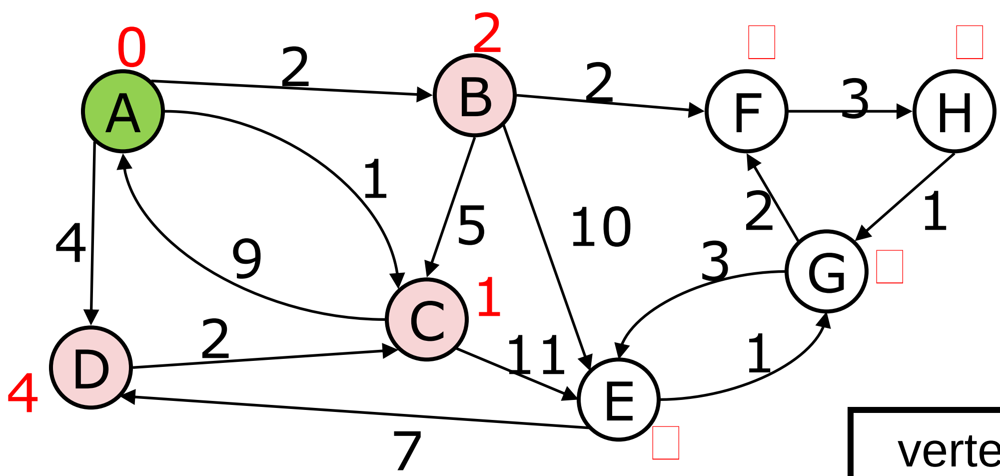
Example #1



vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

<u>Order Added to Known Set:</u>

Example #1

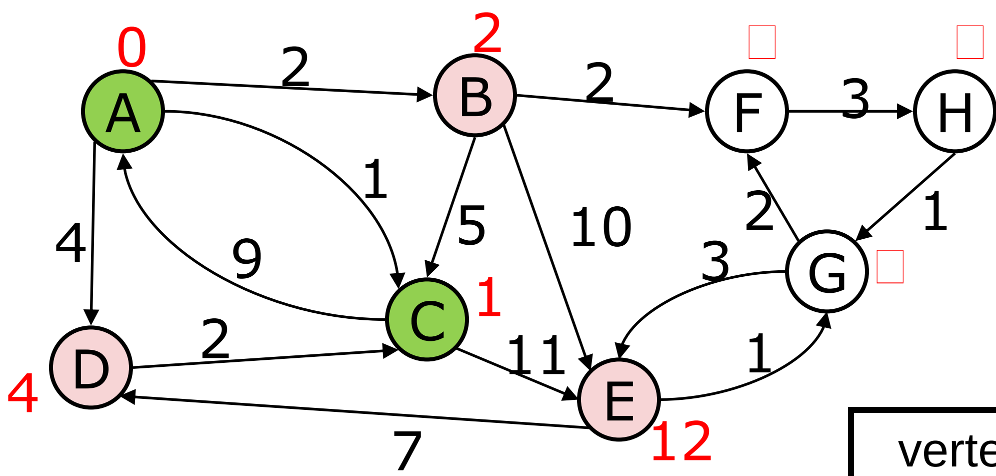


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

A

Example #1

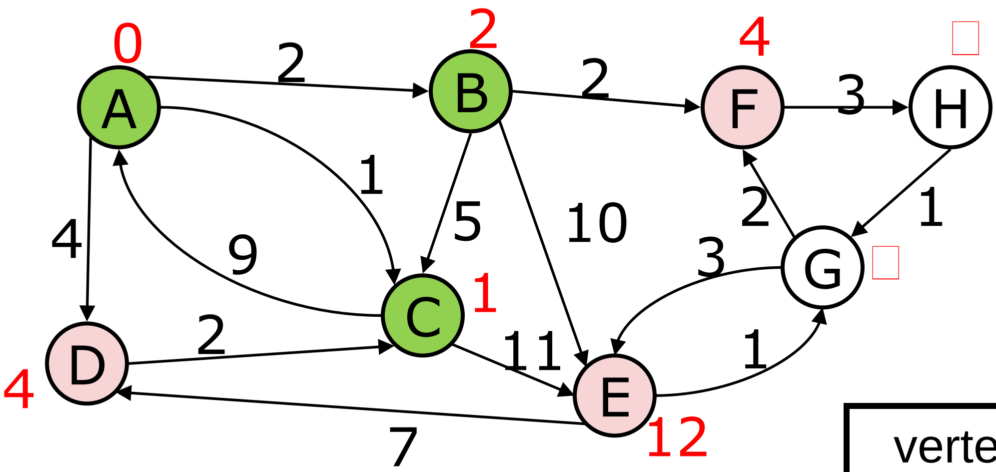


Order Added to Known Set:

A, C

vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Example #1

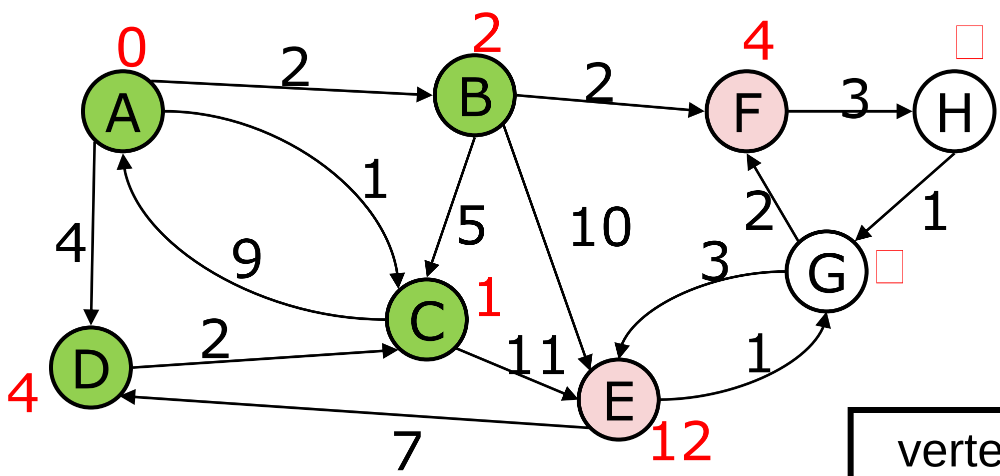


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

<u>Order Added to Known Set:</u>

A, C, B

Example #1

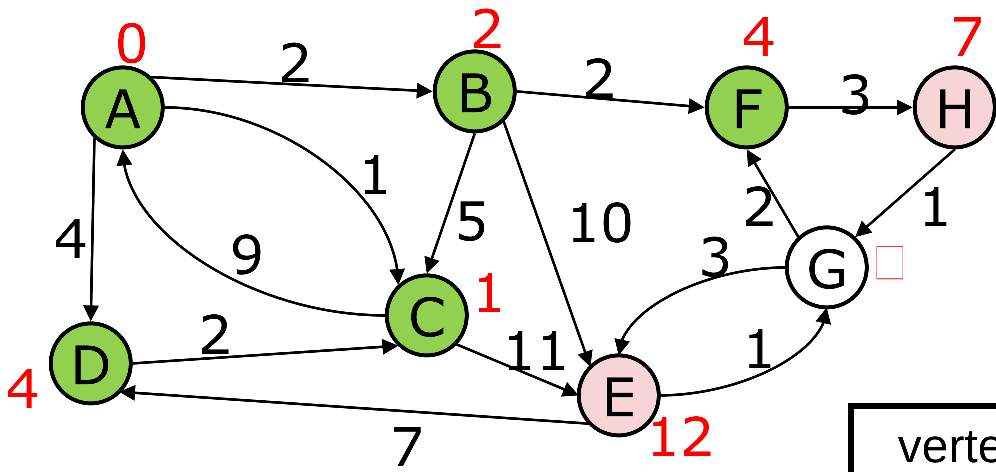


Order Added to Known Set:

A, C, B, D

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Example #1

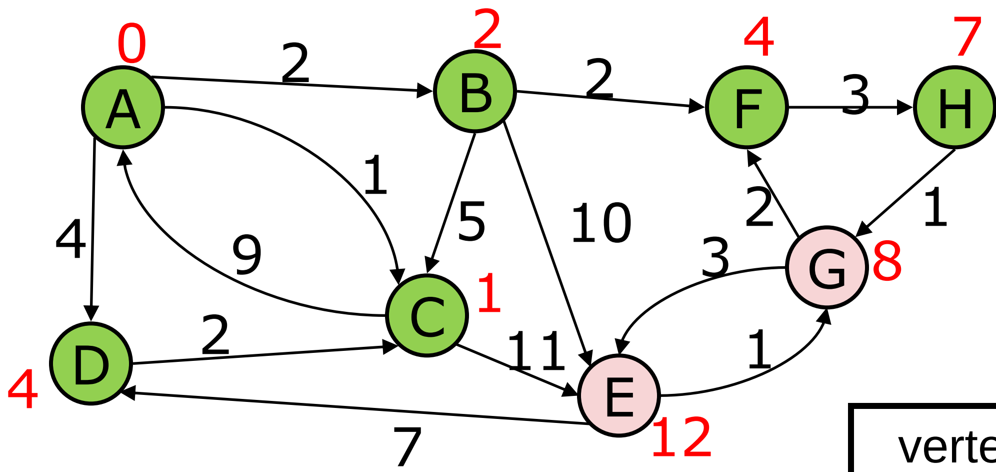


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		$\leq$ 12	C
F	Y	4	B
G		??	
H		$\leq$ 7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F

Example #1

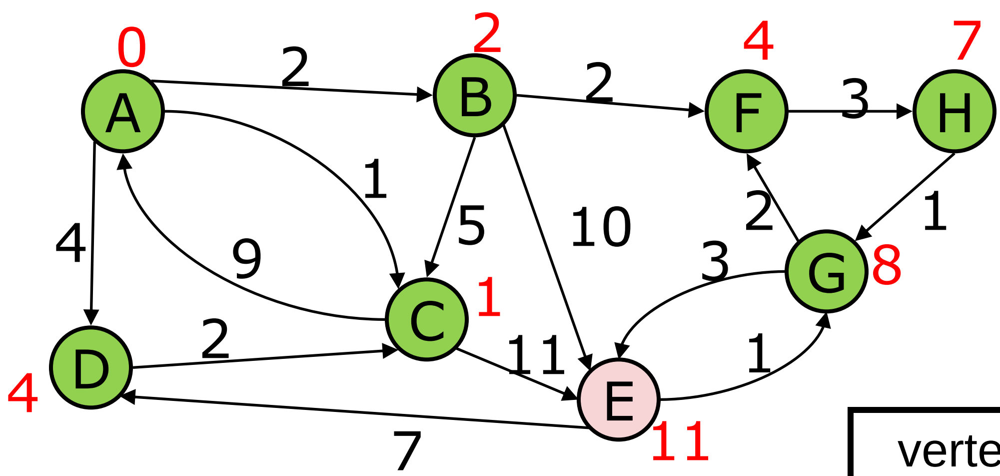


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H

Example #1

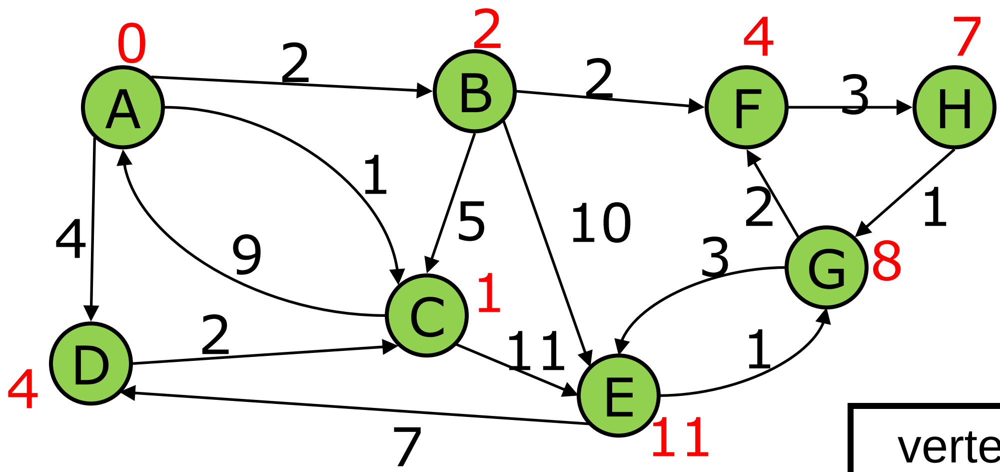


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		$\leq$ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

Important Features

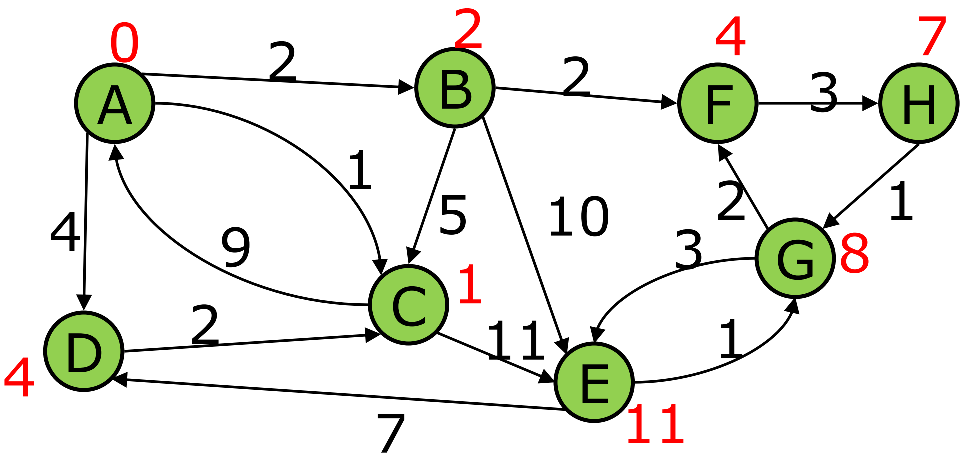
When a vertex is marked known, the cost of the shortest path to that node is known

- The path is also known by following back-pointers

While a vertex is still not known, another shorter path to it might still be found

Interpreting the Results

Now that we're done, how do we get the path from, say, A to E?



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

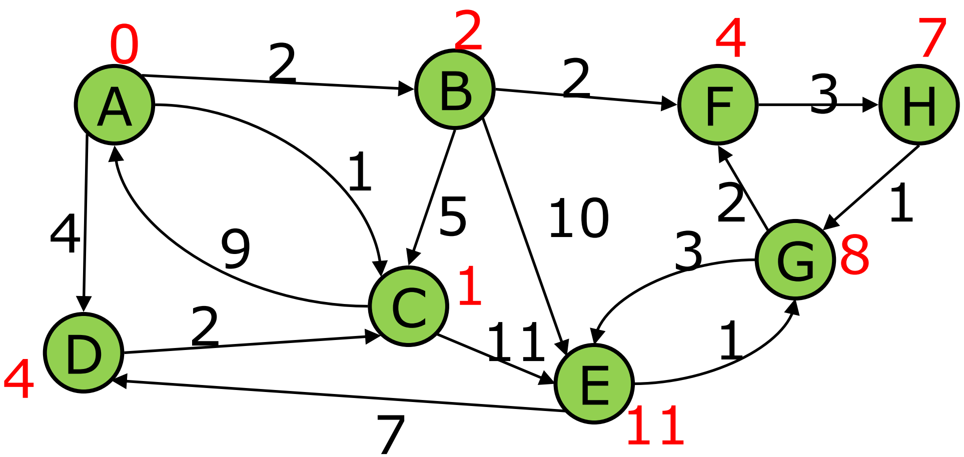
<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

Stopping Short

How would this have worked differently if we were only interested in:

- the path from A to G?
- the path from A to E?

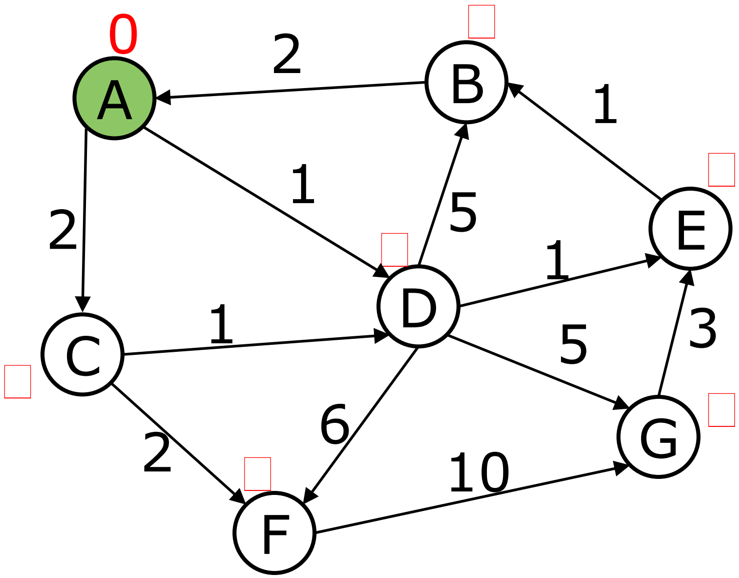


<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

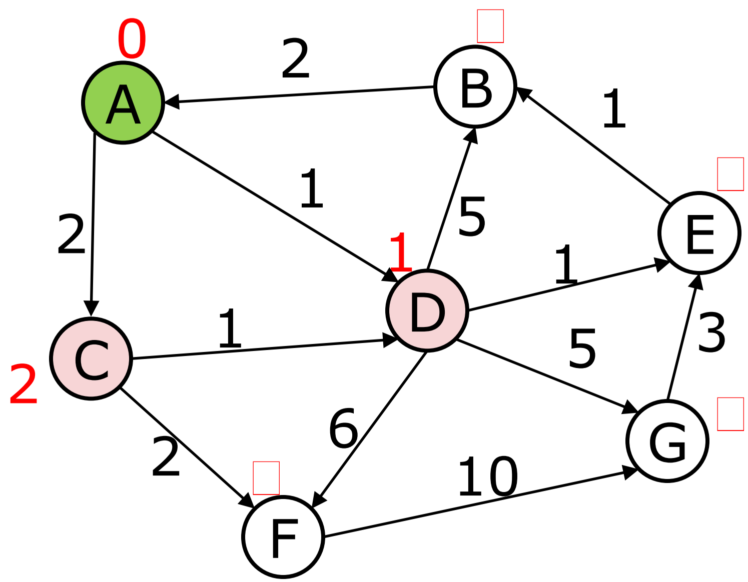
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2

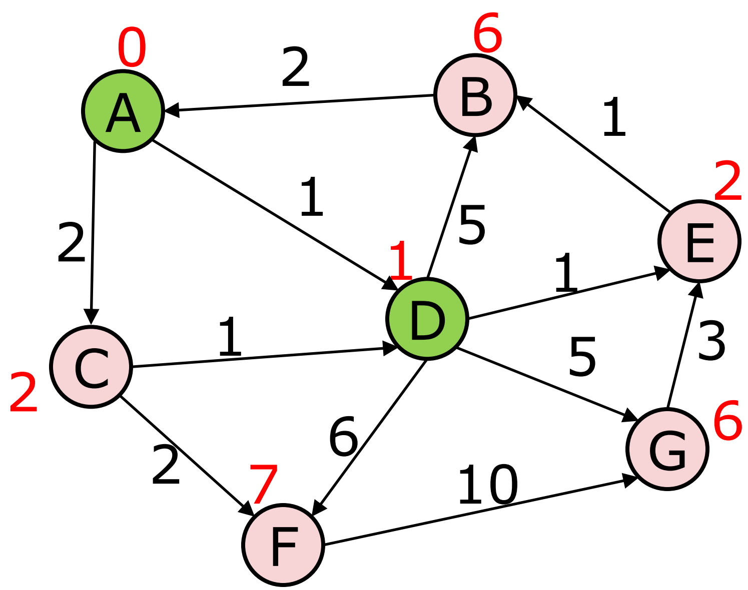


vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

<u>Order Added to Known Set:</u>

A

Example #2

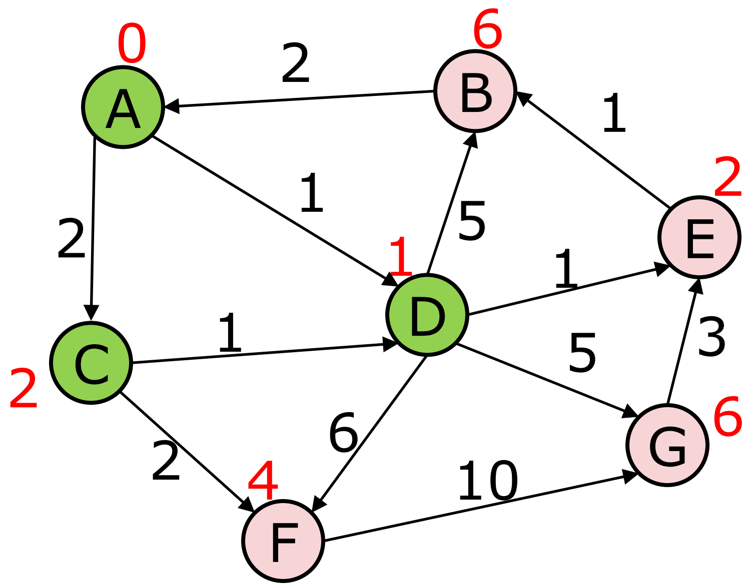


Order Added to Known Set:

A, D

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2

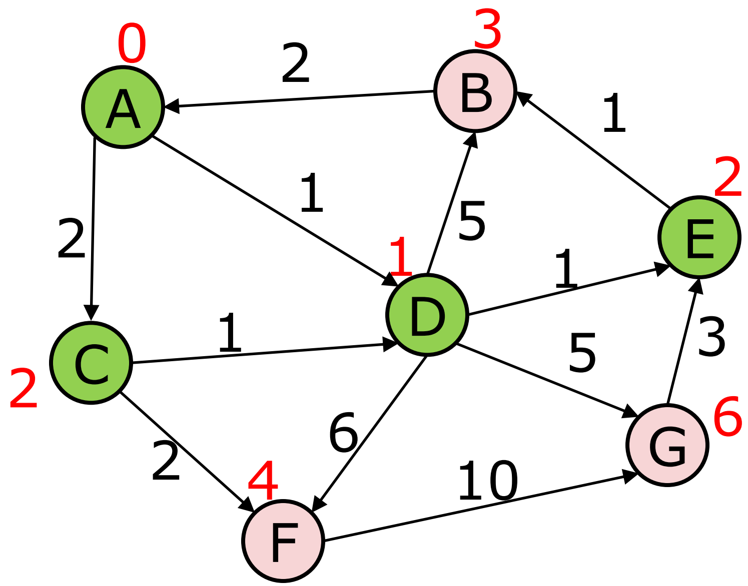


vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C

Example #2

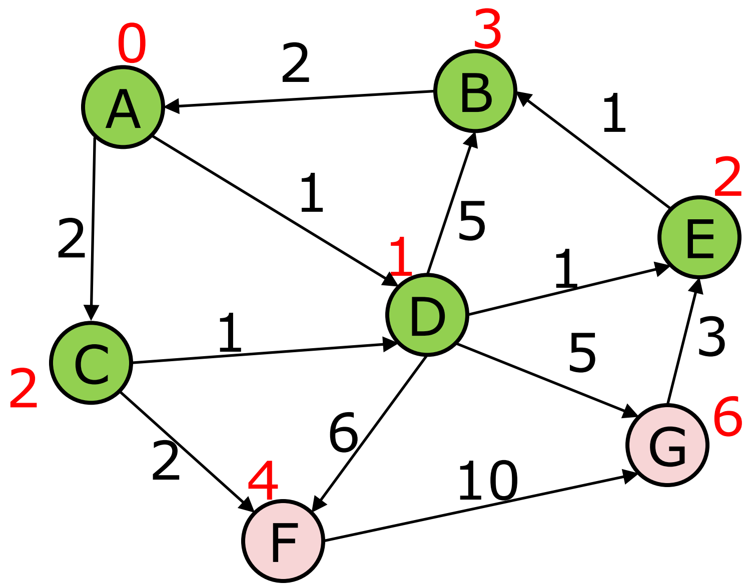


vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Order Added to Known Set:

A, D, C, E

Example #2

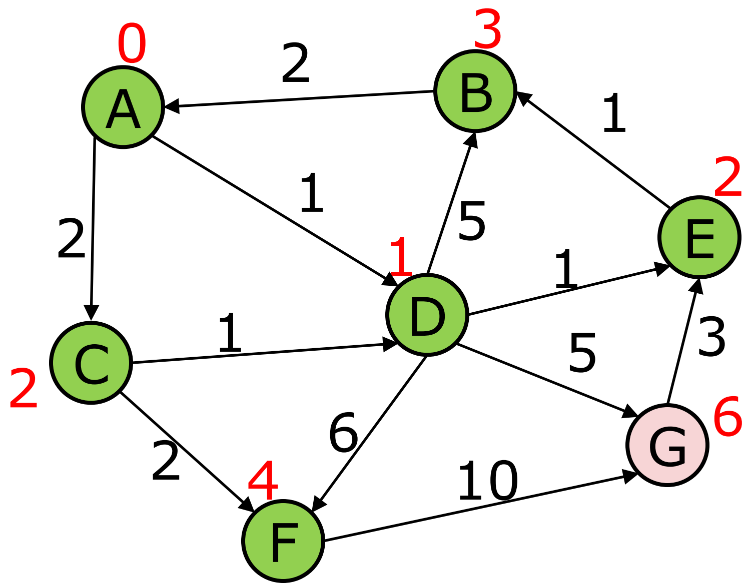


Order Added to Known Set:

A, D, C, E, B

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

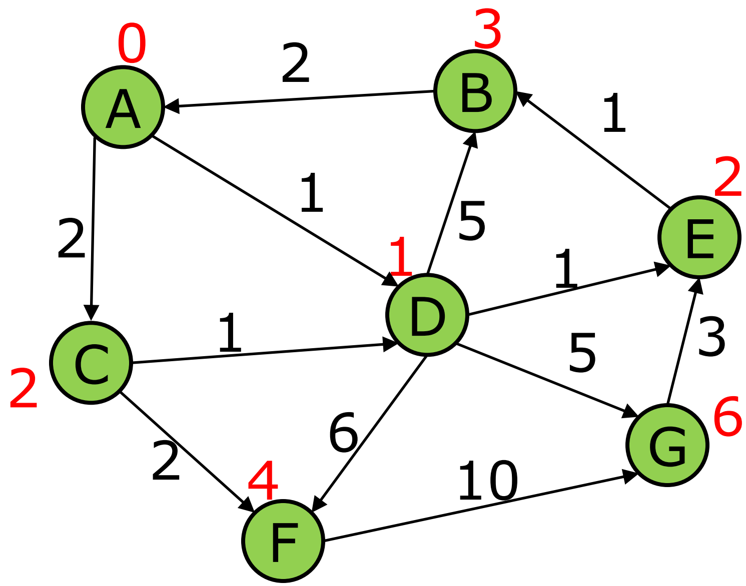


Order Added to Known Set:

A, D, C, E, B, F

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

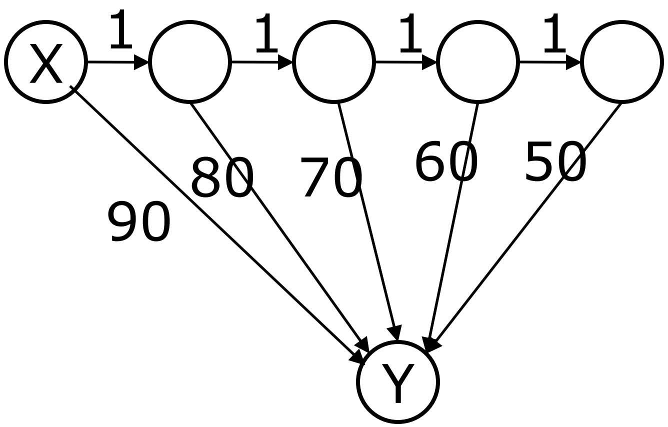


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

<u>Order Added to Known Set:</u>

A, D, C, E, B, F, G

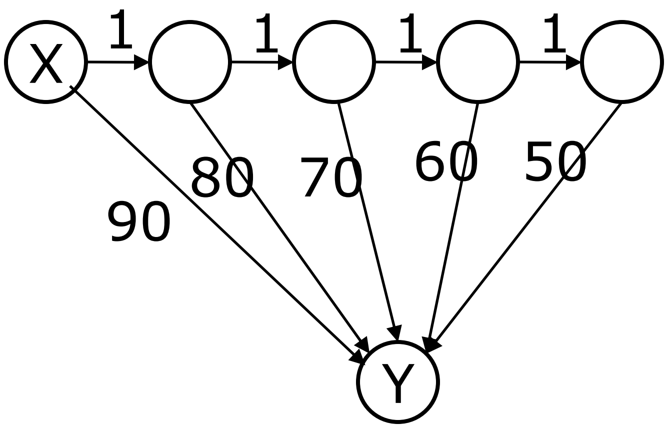
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed?
90, 81, 72, 63, 54

Is this expensive?
No, each *edge* is processed only once

A Greedy Algorithm

Dijkstra's algorithm is an example of a greedy algorithm:

- At each step, irrevocably does what seems best at that step
  - Once a vertex is in the known set, does not go back and readjust its decision
- Locally optimal
  - Does not always mean globally optimal

Where are We?

Have described Dijkstra's algorithm

- For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges

What should we do next?

- Prove the algorithm is correct
- Analyze its efficiency

Correctness: Rough Intuition

All "known" vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node as "known", then by induction this holds and eventually every vertex will be "known"

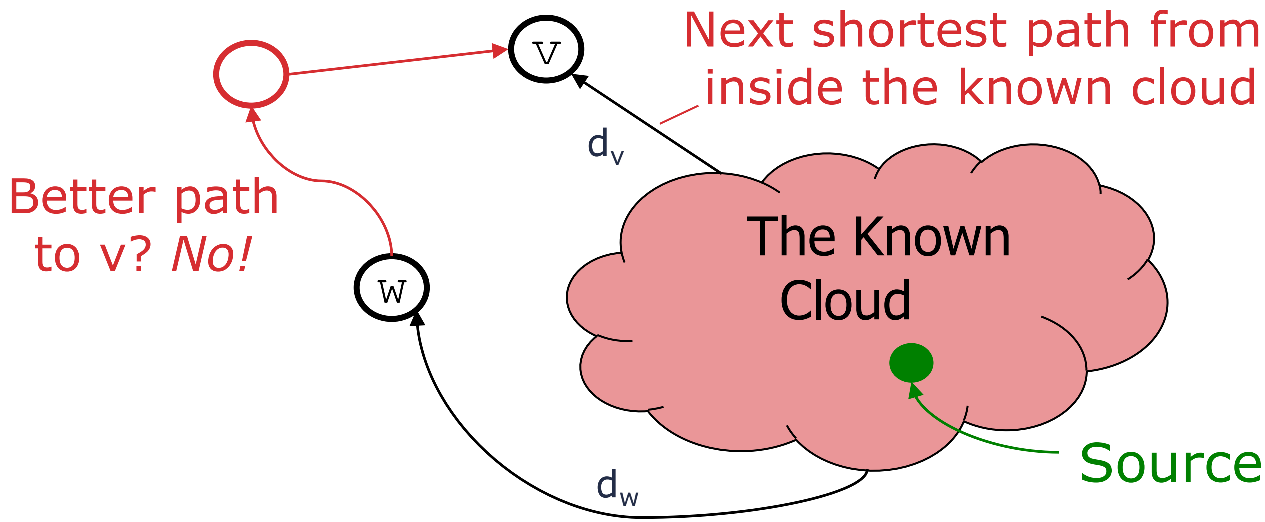
What we need to prove:

- When we mark a vertex as "known", we cannot ever discover a shorter path later in the algorithm
- If we could, then the algorithm fails

How we prove it:

- This holds only because Dijkstra's algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Proof of Correctness (Rough Sketch)



Suppose v is the next node to be marked known ("added to the cloud")

The best-known path to v must have only nodes "in the cloud"

- We have selected it, and we only know about paths through the cloud to a node at the edge of the cloud

Assume the actual shortest path to v is different

- It is not entirely within the cloud, or else we would know about it
- So it must use non-cloud nodes. Let w be the first non-cloud node on this path
- The part of the path up to w is already known and must be shorter than the best-known path to v: $d_w + \ldots < d_v \rightarrow d_w < d_v$
- Ergo, w should have been picked before v. Contradiction.

Efficiency, First Approach

Use pseudocode to determine asymptotic run-time

- Important: note that each edge is processed only once

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  while(not all nodes are known) {
    b = find unknown node with smallest cost
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         a.cost = b.cost + weight((b,a))
         a.path = b
       }
}
```

$O(|V|)$

$O(|V|^2)$

$O(|E|)$

$O(|V|^2)$

Improving Asymptotic Running Time

So far we have an abysmal $O(|V|^2)$

We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next

- We solved it with a queue of zero-degree nodes
- But here we need the lowest-cost node <u>and</u> costs can change as we process edges

Solution?

Improving Asymptotic Running Time

We will us a priority queue

- Hold all unknown nodes
- Priority will be their current cost

But we need to update costs

- Priority queue must have a decreaseKey operation
- For efficiency, each node should maintain a reference from to its position in the queue
  - Eliminates need for O(log n) lookup
  - Conceptually simple, but can be a pain to code up

Efficiency, Second Approach

Use pseudocode to determine asymptotic run-time

- Note that deleteMin() and decreaseKey() operations are independent of each other

```
dijkstra(Graph G, Node start) {
 for each node: x.cost=infinity, x.known=false
 start.cost = 0
 build-heap with all nodes
 while(heap is not empty) {
   b = deleteMin()
   b.known = true
   for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         decreaseKey(a,"new cost - old cost")
          a.path = b
       }
}
```

O(|V|)

O(|V|)

O(|V|log |V|)

O(|E|log|V|)

O(|V|log|V|+|E|log|V|)

Dense versus Sparse Again

First approach: $O(|V|^2)$

Second approach: $O(|V|\log|V|+|E|\log|V|)$

So which is better?

- Sparse: $O(|V|\log|V|+|E|\log|V|)$

  If $|E| = \Theta(|V|)$, then $O(|E|\log|V|)$
- Dense: $O(|V|^2)$

  If $|E| = \Theta(|V|^2)$, then $|E|\log|V| > |V|^2$
- Neither sparse or dense?

  Second approach still likely to be better

But…

Remember these are worst-case and asymptotic

Priority queue might have worse constant factors

On the other hand, for "normal graphs"

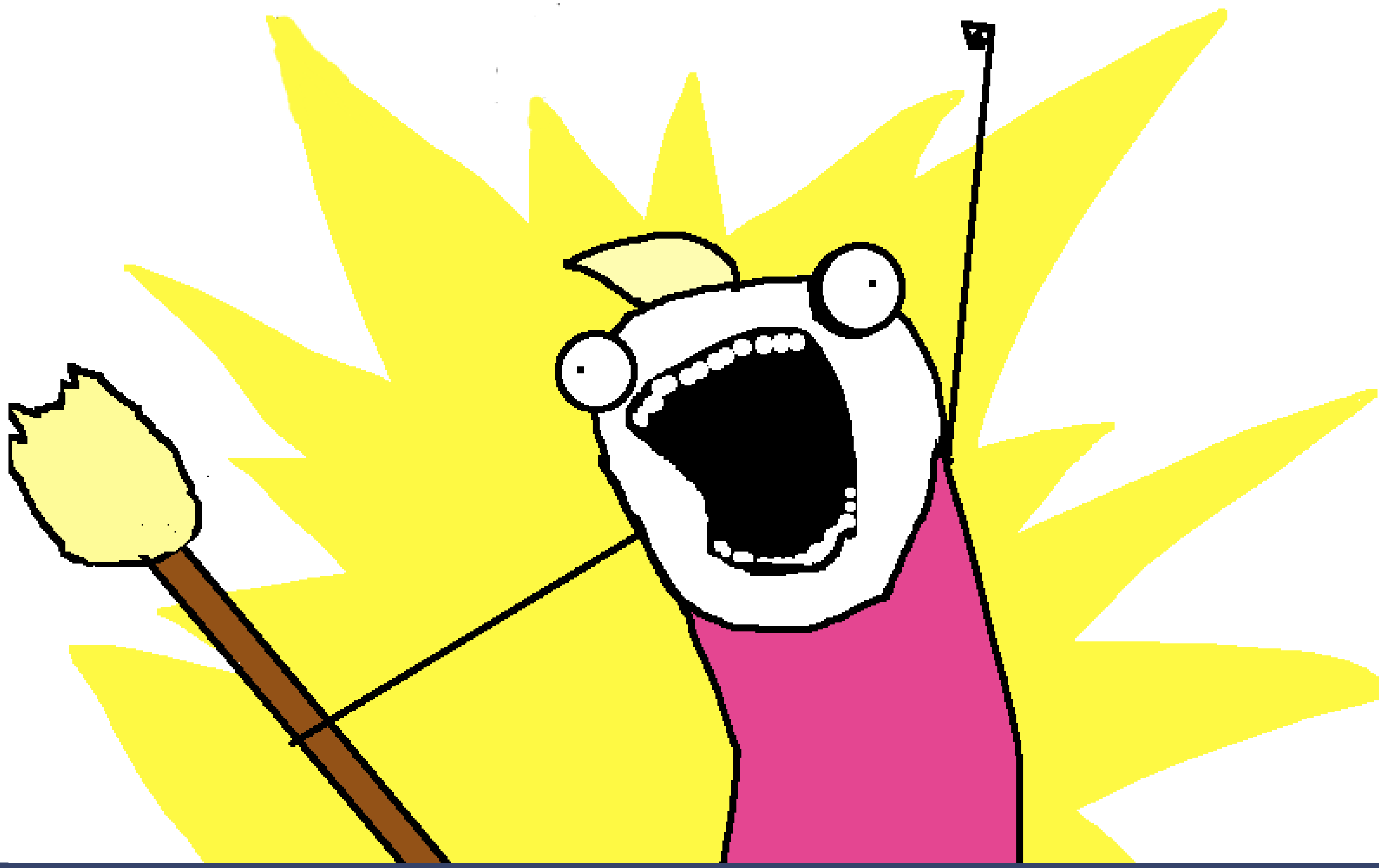
- We might rarely call decreaseKey
- We might not percolate far
- This would make |E|log|V| more like |E|

What about connectedness?

What happens if a graph is disconnected?

Unmarked/unvisited nodes will continue to have a cost of infinity

- Must be careful to do addition correctly: ∞ + (finite value) = ∞
- One speed-up would be to stop once a deleteMin() returns ∞



YOU WANT ALL THE SHORTEST PATHS?

All-Pairs Shortest Path

Dijkstra's algorithm requires a starting vertex

What if you want to find the shortest path between all pairs of vertices in the graph?

- Run Dijkstra's for each vertex v?
- Can we do better? Yep

Dynamic Programming

An algorithmic technique that systematically records the answers to sub-problems in a table and re-uses those recorded results.

Simple Example:

Calculating the Nth Fibonacci number:

$$Fib(N) = Fib(N-1) + Fib(N-2)$$

Recursion would be insanely expensive,

But it is cheap if you already know the results of prior computations

Floyd-Warshall All-Pairs Shortest Path

Dynamic programming algorithm for finding shortest paths between all vertices

Even works for negative edge weights

- Only meaningful in no negative cycles
- Can be used to detect such negative cycles
- Idea: Check to see if there is a path from v to v that has a negative cost

Overall performance:

- Time: $O(|V|^3)$
- Space: $O(|V|^2)$

The Algorithm

M[u][v] stores the cost of the best path from u to v

Initialized to cost of edge between M[u][v]

The algorithm:

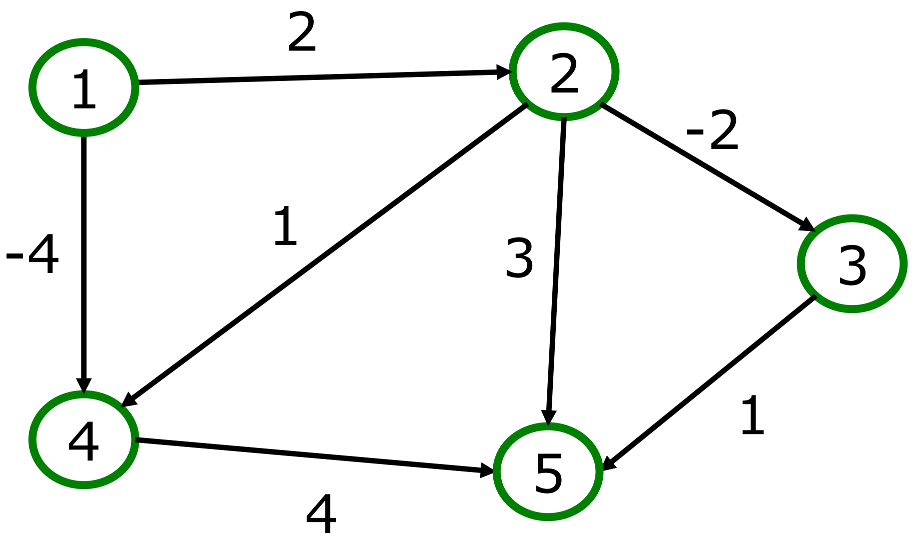
```
for (int k = 1; k =< V; k++)
  for (int i = 1; i =< V; i++)
    for (int j = 1; j =< V; j++)
      if ( M[i][k]+ M[k][j] < M[i][j] )
         M[i][j] = M[i][k]+ M[k][j]
```

Invariant:

After the k^{th} iteration, the matrix M includes the shortest path between all pairs that use on only vertices 1..k as intermediate vertices in the paths

Floydl-Warshall

All-Pairs Shortest Path



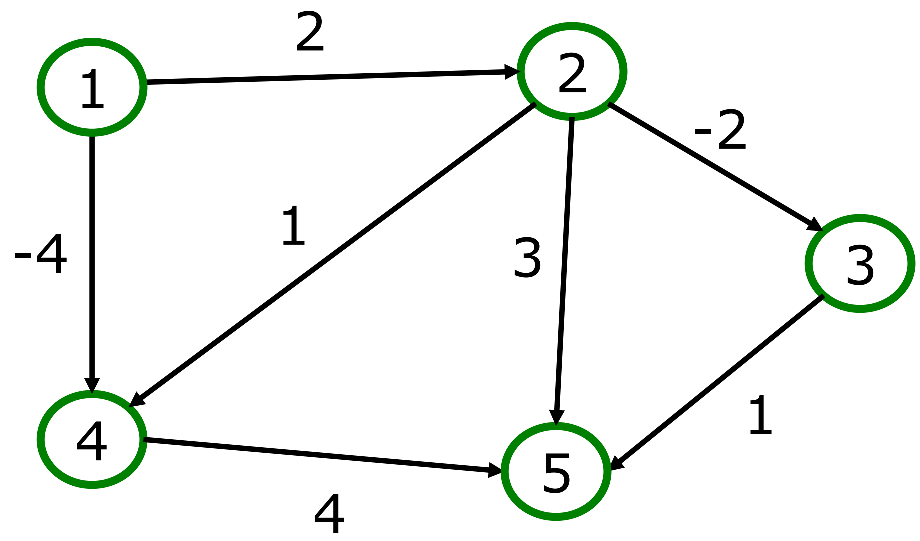
Initial state of the matrix:

	1	2	3	4	5
1	0	2	∞	-4	∞
2	∞	0	-2	1	3
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

Note that non∞edges are indicated in some manner, such as infinity

Floydl-Warshall

All-Pairs Shortest Path



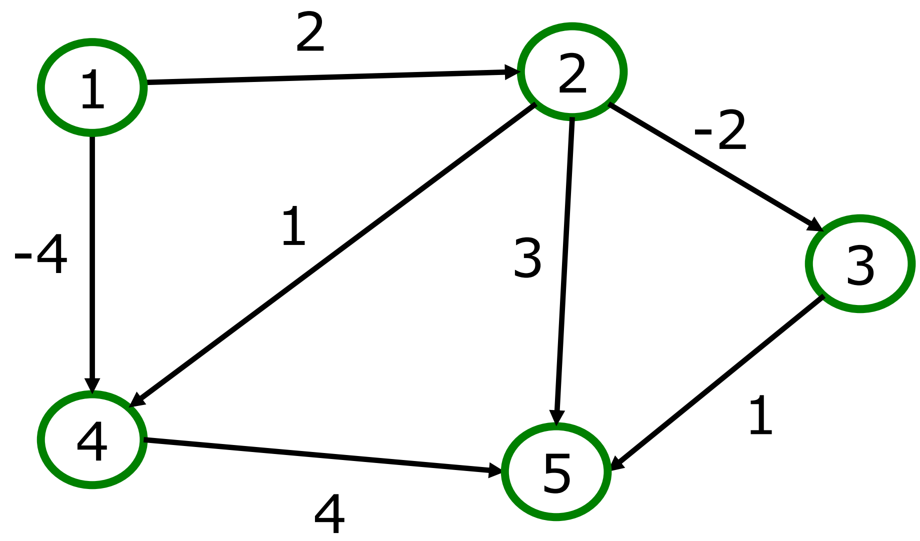
k = 1

	1	2	3	4	5
1	0	2	∞	-4	∞
2	∞	0	-2	1	3
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

M[i][j] =
min(M[i][j], M[i][k]+ M[k][j])

Floydl-Warshall

All-Pairs Shortest Path



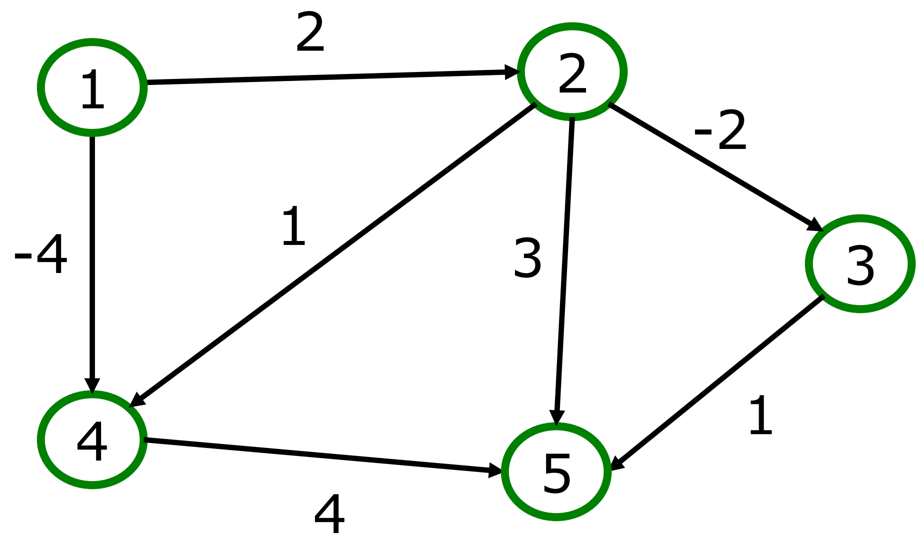
k = 2

	1	2	3	4	5
1	0	2	0	-4	5
2	∞	0	-2	1	3
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

$$M[i][j] = \min(M[i][j],\ M[i][k] + M[k][j])$$

Floydl-Warshall

All-Pairs Shortest Path



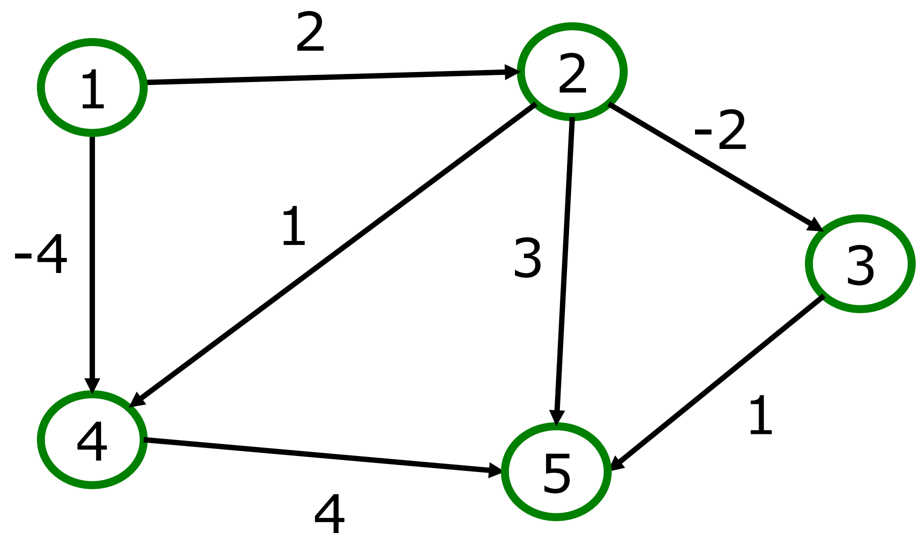
k = 3

	1	2	3	4	5
1	0	2	0	-4	1
2	∞	0	-2	1	-1
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

M[i][j] =
min(M[i][j], M[i][k]+ M[k][j])

Floydl-Warshall

All-Pairs Shortest Path



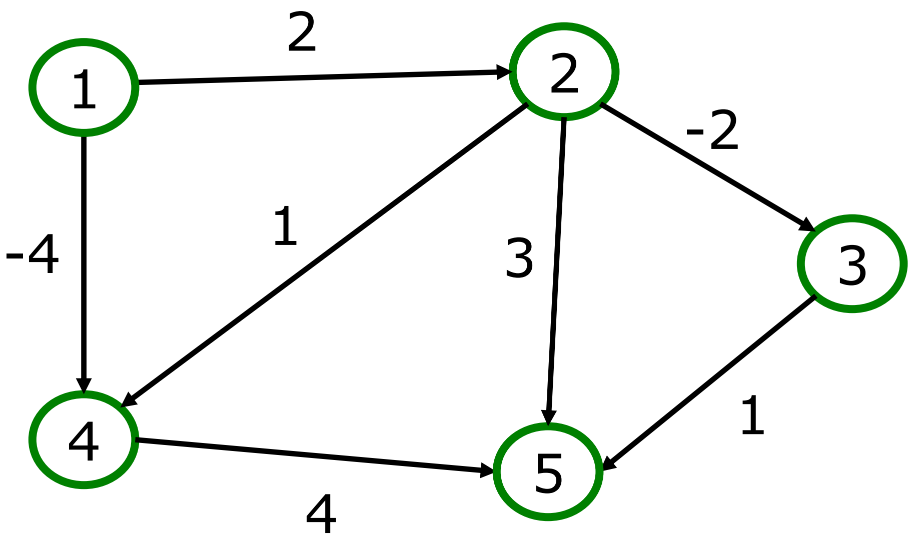
k = 4

	1	2	3	4	5
1	0	2	0	-4	0
2	∞	0	-2	1	-1
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

M[i][j] =
min(M[i][j], M[i][k]+ M[k][j])

Floydl-Warshall

All-Pairs Shortest Path



k = 5

	1	2	3	4	5
1	0	2	0	-4	0
2	∞	0	-2	1	-1
3	∞	∞	0	∞	1
4	∞	∞	∞	0	4
5	∞	∞	∞	∞	0

M[i][j] =
min(M[i][j], M[i][k]+ M[k][j])

What about connectedness?

What happens if a graph is disconnected?

Floyd-Warshall will still calculate all-pair shortest paths.

Some will remain ∞ to indicate that no path exists between those vertices

What Comes Next?

In the logical course progression, we would study the next graph topic:

Minimum Spanning Trees

They are trees… that span… minimally!! Woo!!

But alas, we need to align lectures with projects and homework, so we will instead

- Start parallelism and concurrency
- Come back to graphs at the end of the course