# CSE 332 Autumn 2023
# Lecture 7: Priority Queues & Recurrences

Nathan Brunelle

http://www.cs.uw.edu/332

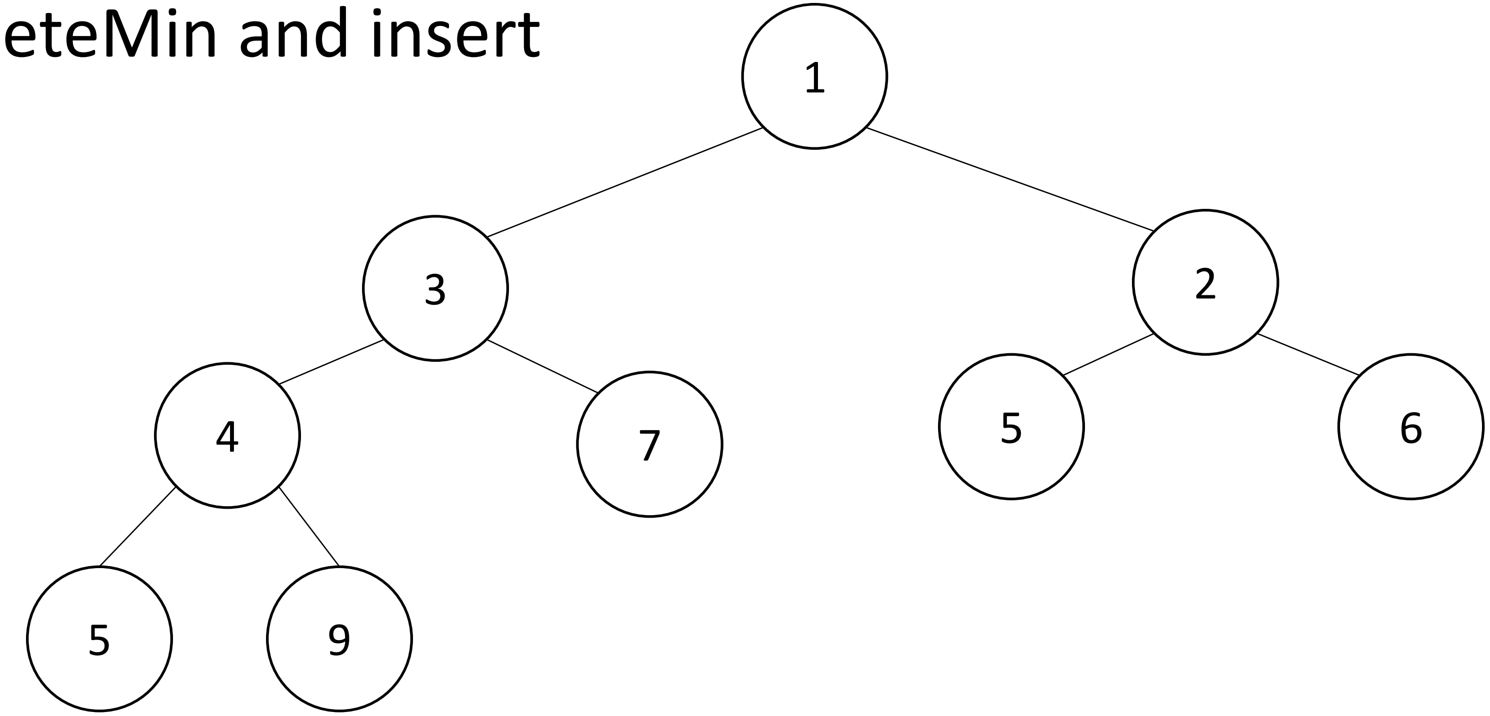
# Thinking through implementations

| Data Structure | Worst case time to insert | Worst case time to deleteMin |
|---|---|---|
| Unsorted Array | $\Theta(1)$ | $\Theta(n)$ |
| Unsorted Linked List | $\Theta(1)$ | $\Theta(n)$ |
| Sorted Circular Array | $\Theta(n)$ | $\Theta(n)$ |
| Sorted Linked List | $\Theta(n)$ | $\Theta(1)$ |
| Binary Search Tree | $\Theta(n)$ | $\Theta(1)$ |
| Binary Heap | $\Theta(\log n)$ | $\Theta(\log n)$ |

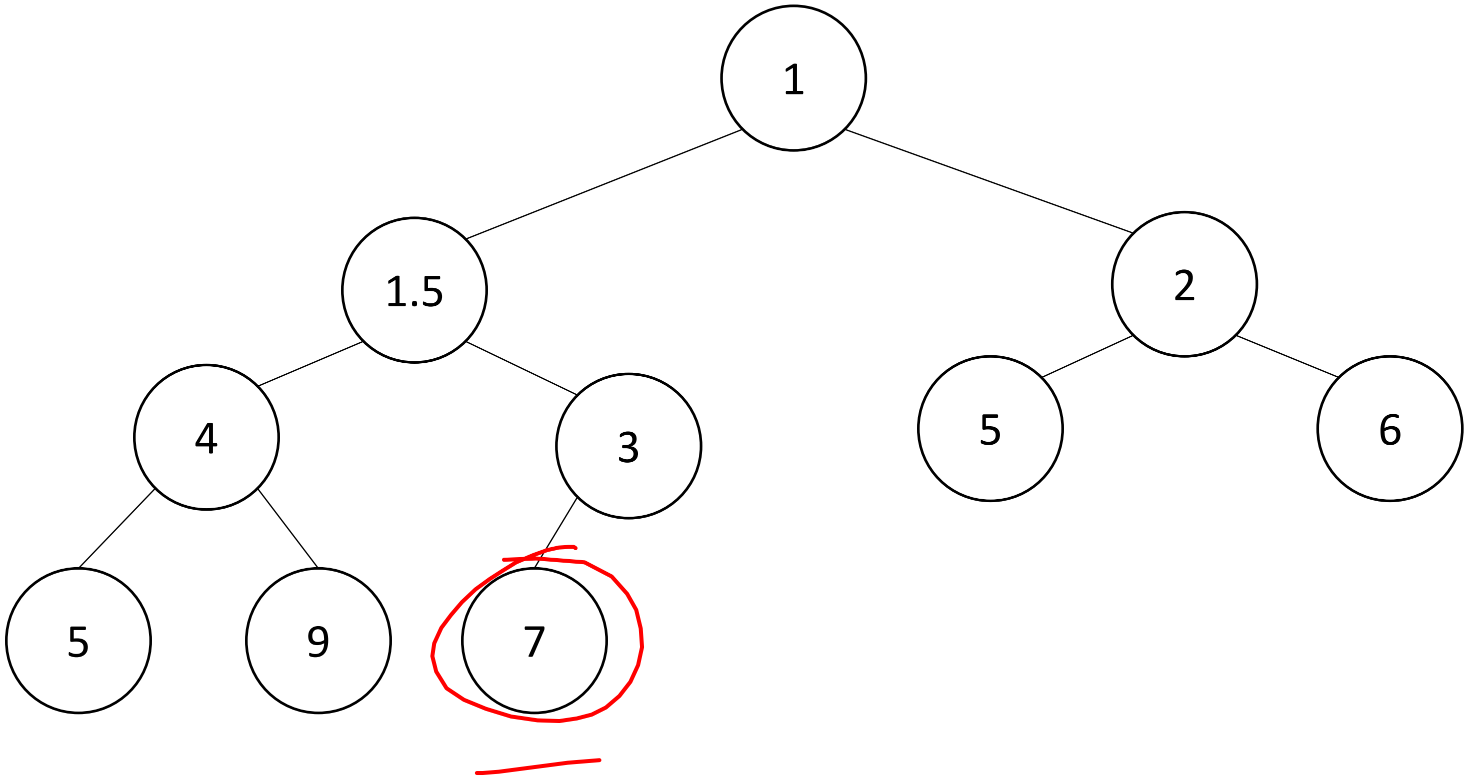
Note: Assume we know the maximum size of the PQ in advance

# Heap – Priority Queue Data Structure

- Idea: We need to keep some ordering, but it doesn't need to be perfectly sorted
- $\Theta(\log n)$ worst case for deleteMin and insert

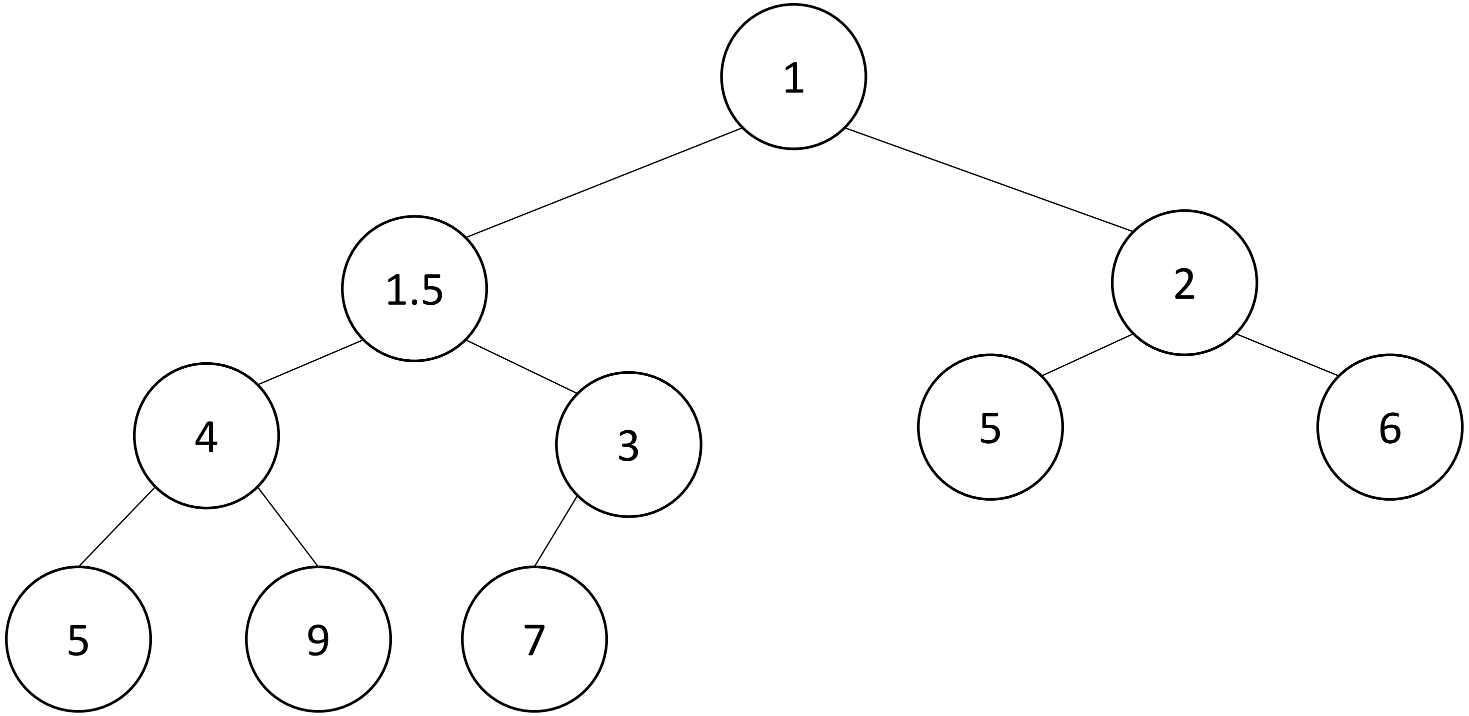


# Heap Insert




```
insert(item){
    put item in the "next open" spot (keep tree complete)
    while (item.priority < parent(item).priority){
        swap item with parent
    }
}
```

# Heap deleteMin




```
deleteMin(){
    min = root
    br = bottom-right item
    move br to the root
    while(br > either of its children){
        swap br with its smallest child
    }
    return min
}
```

down

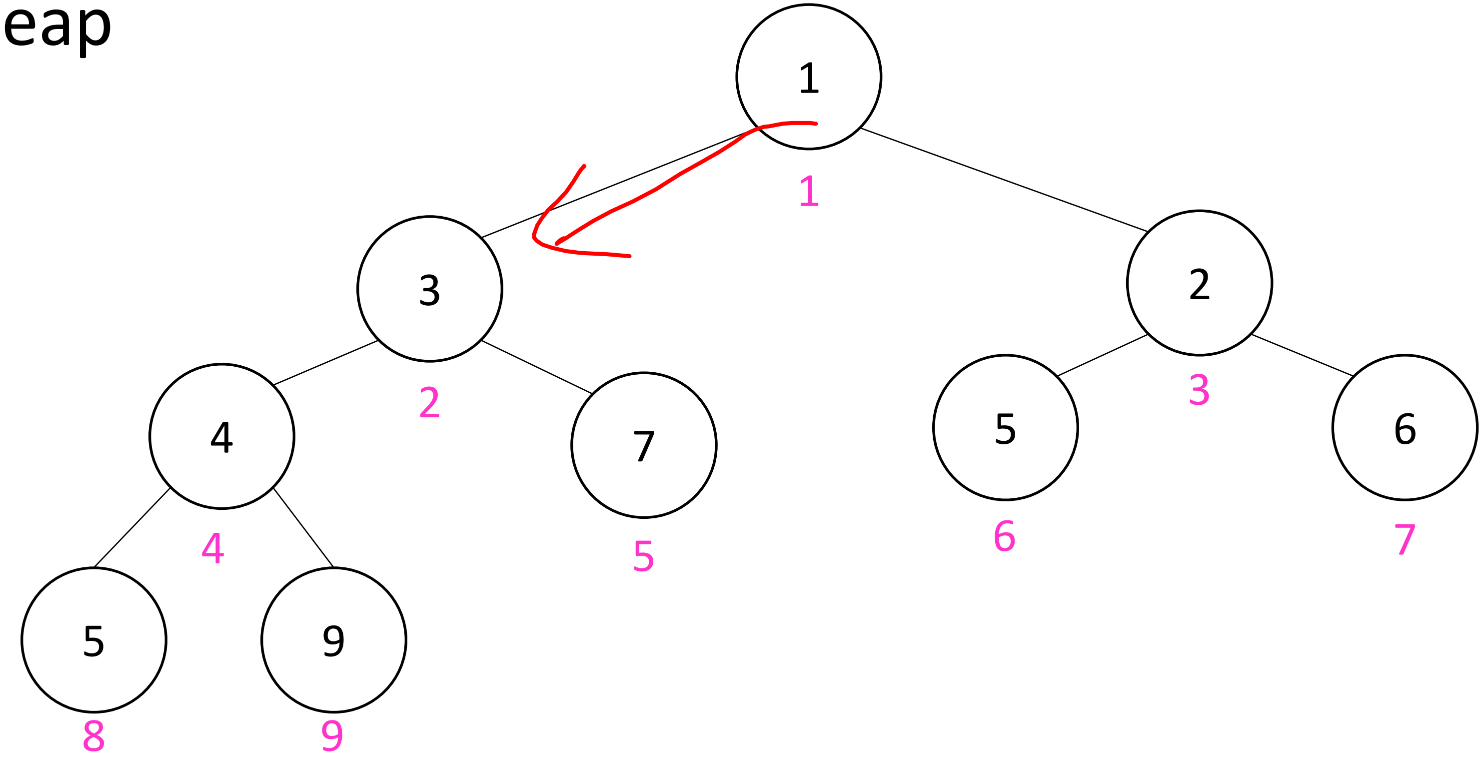
# Percolate Up and Down

- Goal: restore the “Heap Property”
- Percolate Up:
  - Take a node that may be smaller than a parent, repeatedly swap with a parent until it is larger than its parent
- Percolate Down:
  - Take a node that may be larger than one of its children, repeatedly swap with smallest child until both children are larger
- Worst case running time of each:
  - $\Theta(\log n)$

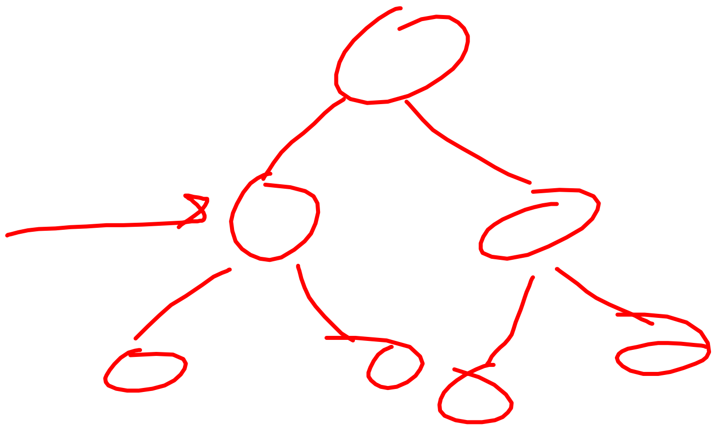
# Representing a Heap

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 3 | 2 | 4 | 7 | 5 | 6 | 5 | 9 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

- Every complete binary tree with the same number of nodes uses the same positions and edges
- Use an array to represent the heap
- Index of root: 1
- Parent of node $i$: $\left\lfloor \frac{i}{2} \right\rfloor$
- Left child of node $i$: $2 \cdot i$
- Right child of node $i$: $2 \cdot i + 1$
- Location of the leaves: last half



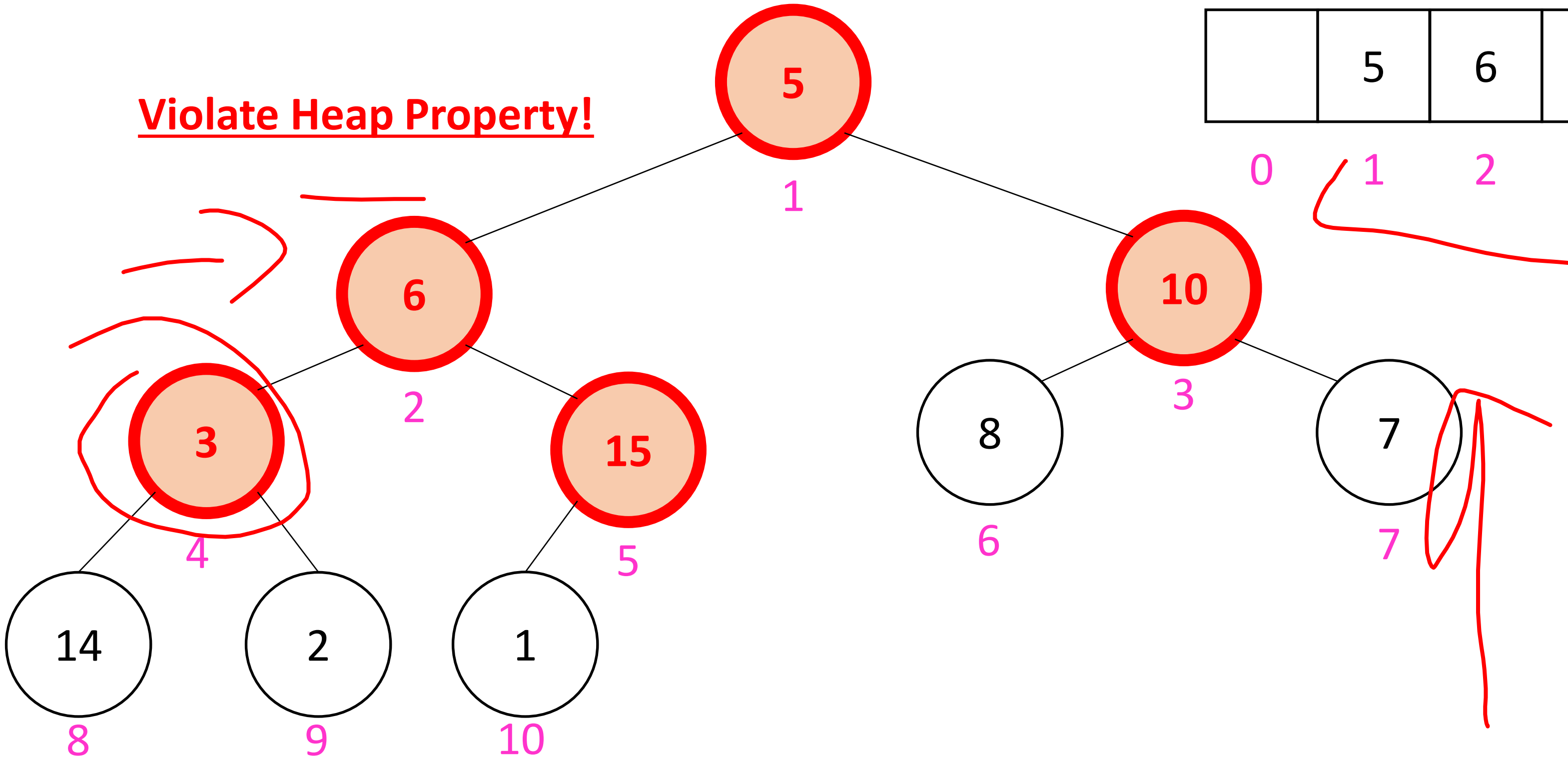
# Other Operations



- Increase Key
    - Given the index of an item in the PQ, subtract from its priority value
    - Update the priority, then percolate [up or down?]
- Decrease Key
    - Given the index of an item in the PQ, add to its priority value
    - Update the priority, then percolate [up or down?]
- Remove
    - Given the item at the given index from the PQ
    - Change its priority to $-\infty$
    - deleteMin

# Building a Heap From "Scratch"

- Suppose we had $n$ items and wanted to "heapify" them




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 6 | 10 | 3 | 15 | 8 | 7 | 14 | 2 | 1 |
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Two ways to "fix" the heap:

1) Percolate Up
2) Percolate Down

# Floyd’s buildHeap method

- Working towards the root, one row at a time, percolate down

```
buildHeap(){
    for(int i = size; i>0; i--){
        percolateDown(i);
    }
}
```

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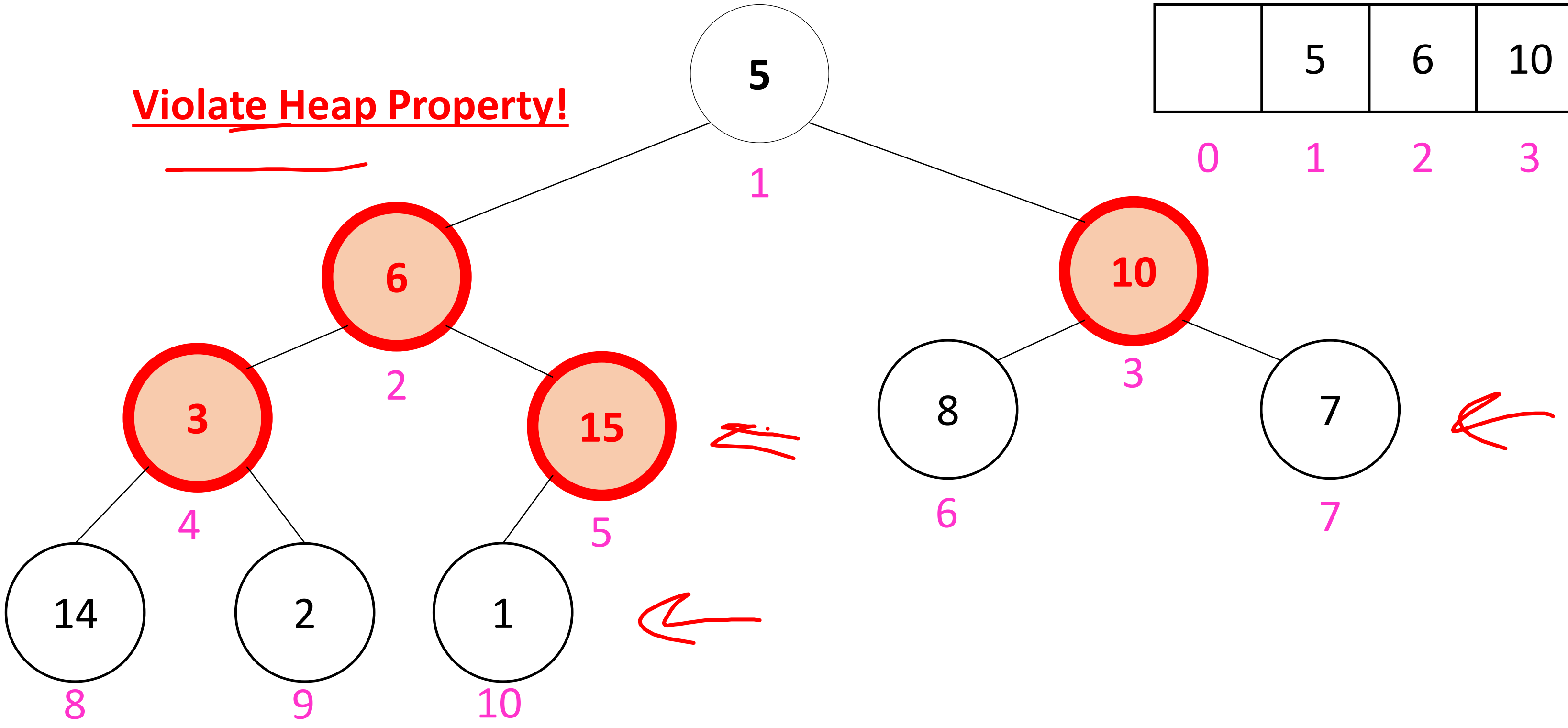


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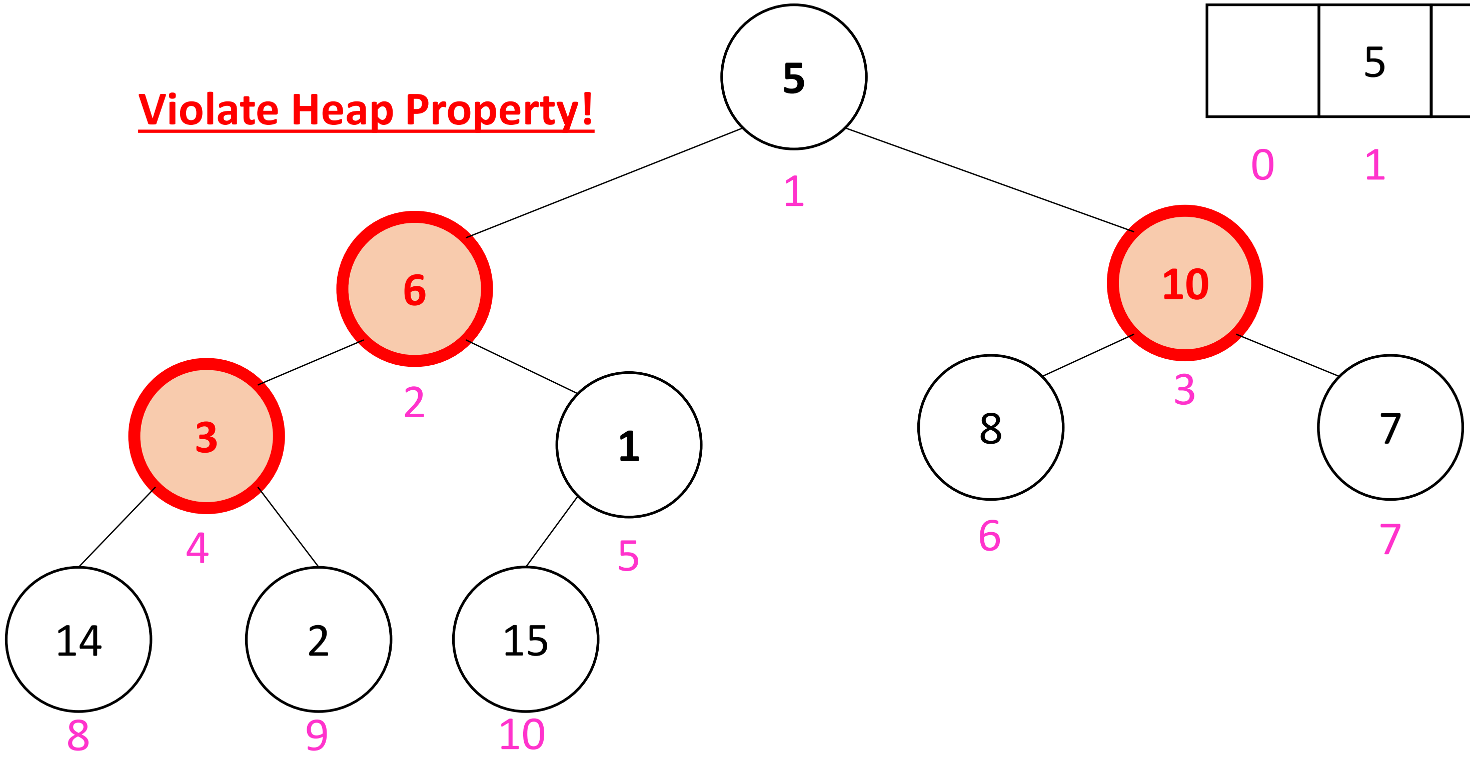


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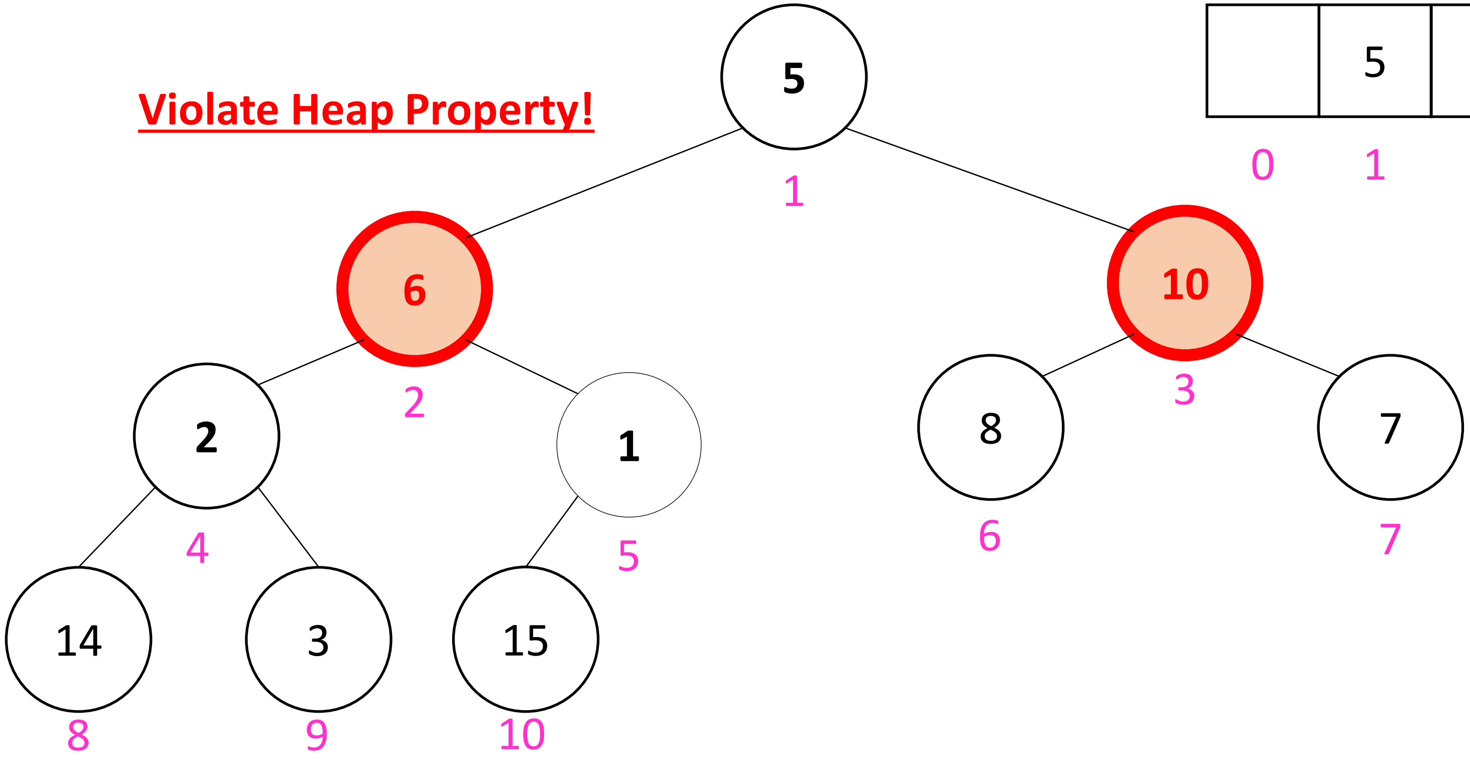


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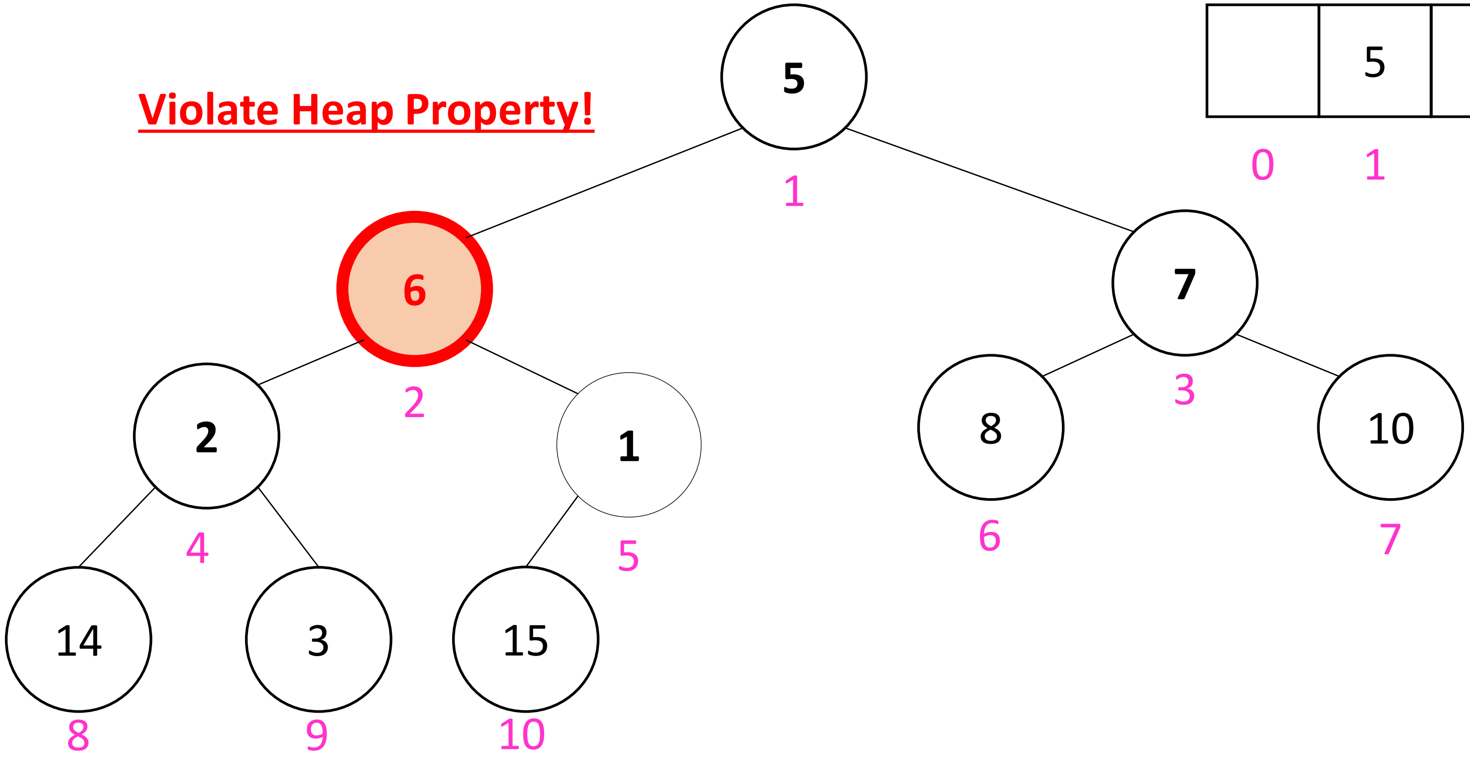


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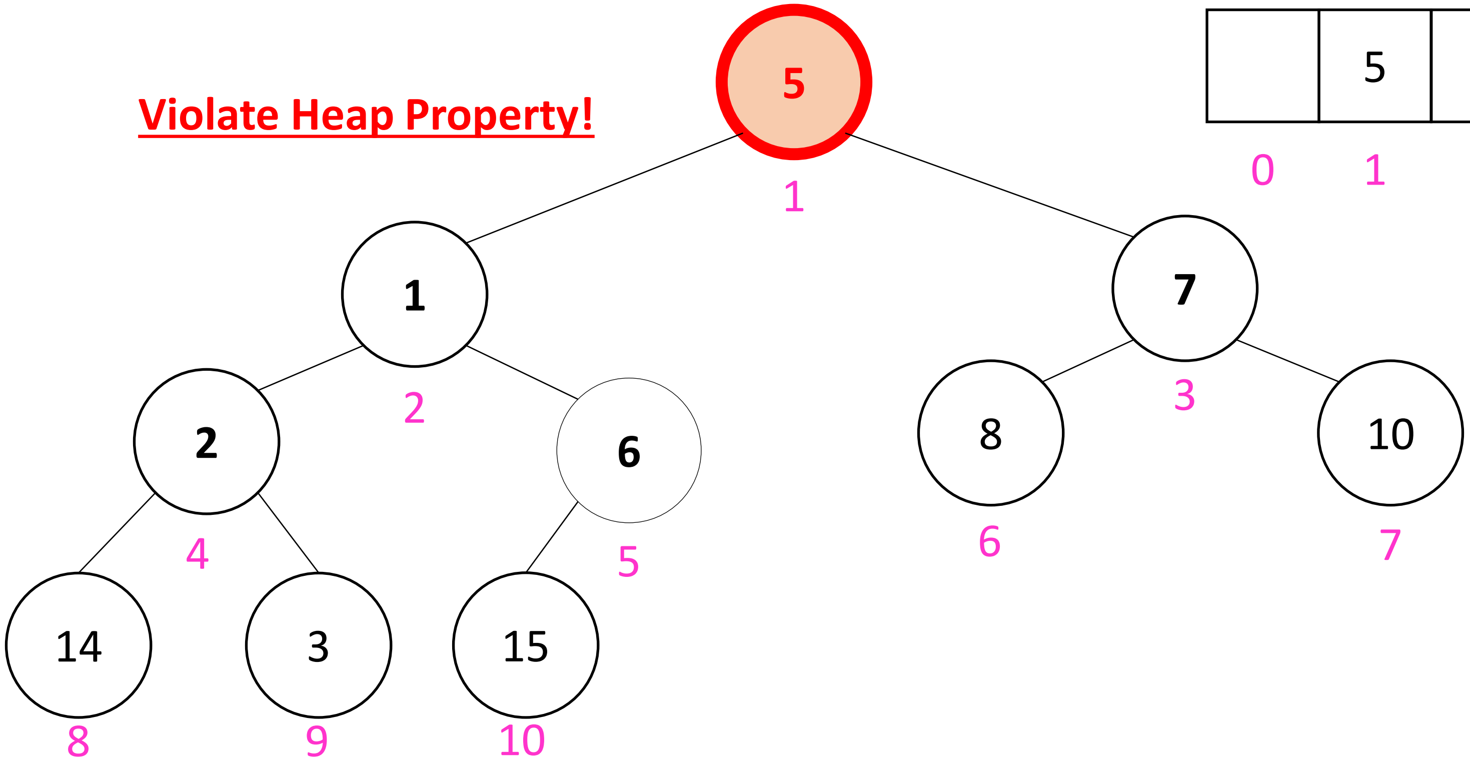


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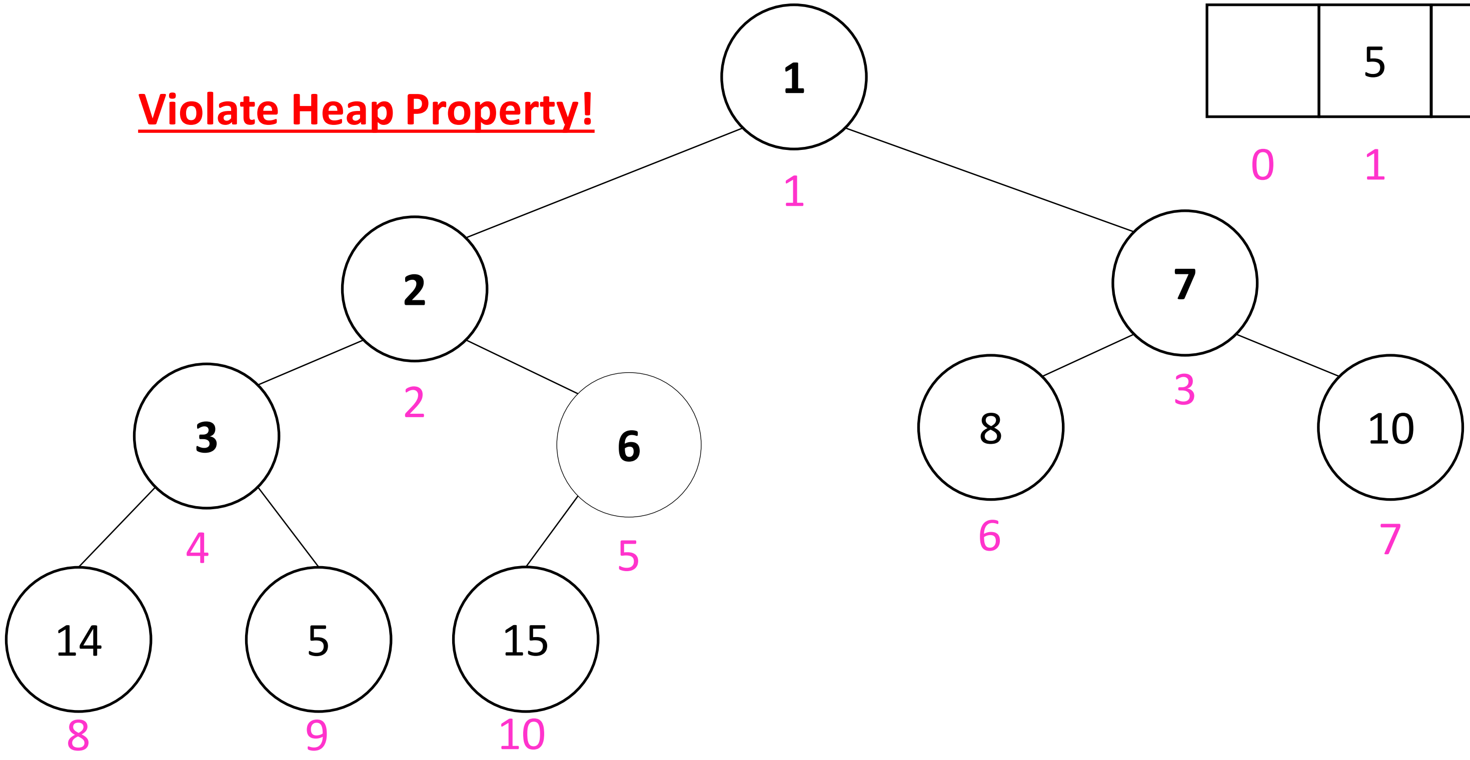


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```
buildHeap(){
    for(int i = size; i>0; i--){
        percolateDown(i);
    }
}
```

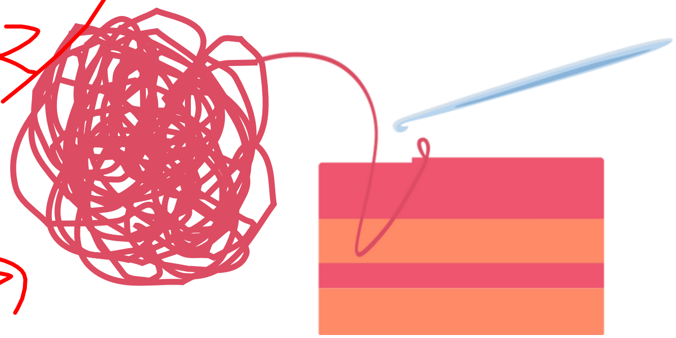
# How long did this take?

```
buildHeap(){
  for(int i = size; i>0; i--){
    percolateDown(i);
  }
}
```

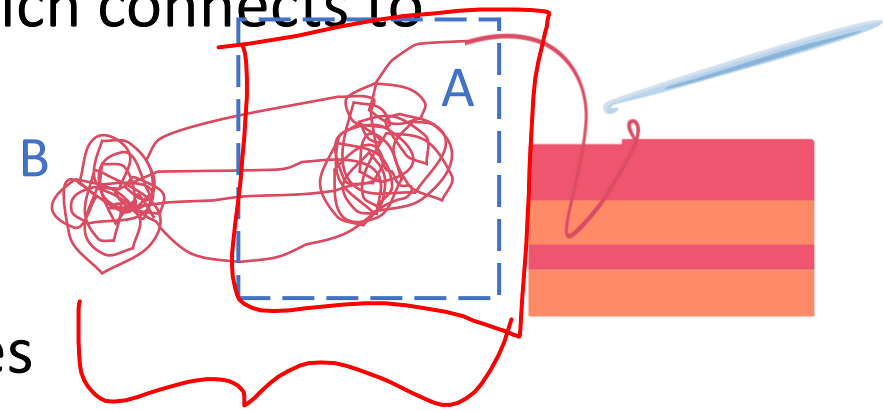
- Worst case running time of buildHeap:
- No node can percolate down more than the height of its subtree
  - When i is a leaf: 0
  - When i is second-from-last level: 1
  - When i is third-from-last level: 2
- Overall Running time:
  - $\frac{n}{2}$ of the items are leaves
    - 0 swaps total
  - $\frac{n}{4}$ of the items are at second-from-last level
    - $\frac{n}{4}$ total swaps
  - $\frac{n}{8}$ of the items are at third-from-last level
    - $\frac{n}{8} * 2$ total swaps
  - $\frac{n}{16} * 3$ total swaps
  - This sum converges to $2n \in \Theta(n)$

→ $T(n) = f(n) + T(\frac{n}{2})$

# End-of-Yarn Finding



1. Set aside the already-obtained "beginning"
2. If you see the end of the yarn, you're done!
3. Separate the pile of yarn into 2 piles, note which connects to the beginning (call it pile A, the other pile B)




Repeat on pile with end

4. Count the number of strands crossing the piles
5. If the count is even, pile A contains the end, else pile B does

# Analysis of Recursive Algorithms

- Overall structure of recursion:
  - Do some non-recursive "work"
  - Do one or more recursive calls on some portion of your input
  - Do some more non-recursive "work"
  - Repeat until you reach a base case
- Running time: $T(n) = T(p_1) + T(p_2) + \cdots + T(p_x) + f(n)$
  - The time it takes to run the algorithm on an input of size $n$ is:
  - The sum of how long it takes to run the same algorithm on each smaller input
  - Plus the total amount of non-recursive work done at that step
- Usually:
  - $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$
    - Called "divide and conquer"
  - $T(n) = T(n - c) + f(n)$
    - Called "chip and conquer"

# How Efficient Is It?

- $T(n) = count(n) + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$
- $T(n) = 5 + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$
- Base case: $T(1) = 5$

$T(n) =$ "cost" of running the entire algorithm on an $n$ inch string

$count(n) =$ "cost" of counting the crossing strands (I arbitrarily picked 5)

# Let's Solve the Recurrence!

$T(1) = 5$

$T(n) = 5 + T(n/2)$

$5 + T(n/4)$

$5 + T(n/8)$

$\dots$

$5$

$\lceil \log_2 n \rceil$

$$T(n) = \sum_{i=1}^{\lceil \log_2 n \rceil} 5 = 5 \lceil \log_2 n \rceil$$

$$T(n) \in \Theta(\log n)$$

# Recursive Linear Search

```
search(value, list){
        if(list.isEmpty()){
                return false;
        {
        if (value == list[0]){
                return true;
        }
        list.remove(0);
        return search(value, list);
}
```

$$T(n) = T(n-1) + 7$$

# Unrolling Method

- Repeatedly substitute the recursive part of the recurrence
- $T(n) = T(n-1) + c$
- $T(n) = T(n-2) + c + c$
- $T(n) = T(n-3) + c + c + c$
- …
- $T(n) = c + c + c + \cdots + c$
  - How many $c$'s?

# Recursive List Summation

$T(n) = 2T(\frac{n}{2}) + c$

```
sum(list){
        return sum_helper(list, 0, list.size);
}
sum_helper(list, low, high){
        if (low == high){ return 0; }
        if (low == high-1){ return list[low]; }
        middle = (high+low)/2;
        return sum_helper(list, low, middle) + sum_helper(list, middle, high);
}
```

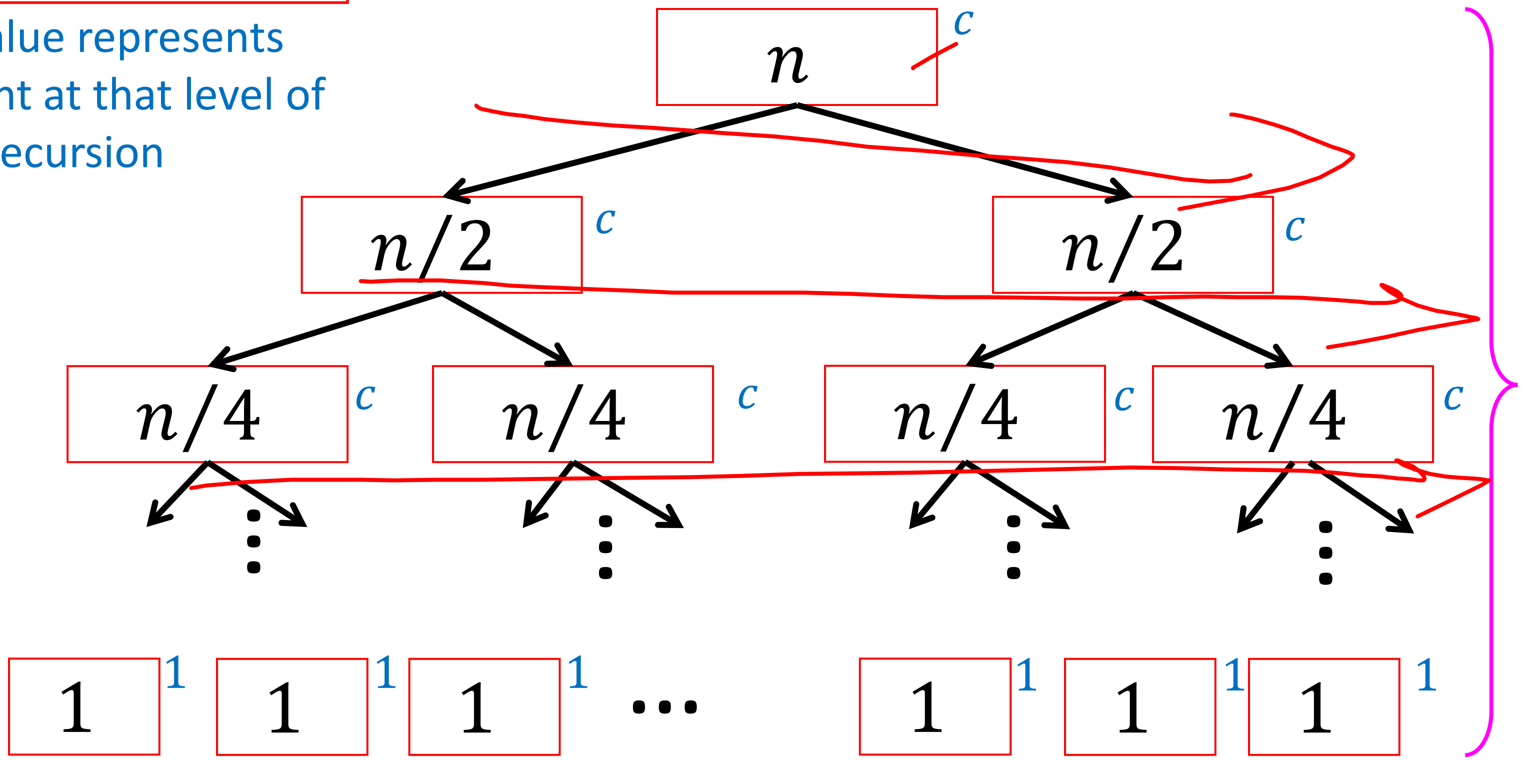
1 2

# Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + c$$




$\Rightarrow 2^i \cdot c$ work per level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} 2^i \cdot c$$

# Recursive List Summation

$$T(n) = \sum_{i=1}^{\log_2 n} 2^i \cdot c$$

$$= c \cdot \sum_{i=1}^{\log_2 n} 2^i$$

$$= c\left(\frac{1 - 2^{\log_2 n}}{1 - 2}\right)$$

# Binary Search

```
search(value, sortedArr){
	return helper(value, sortedArr, 0, sortedArr.length);
}
helper(value, arr, low, high){
	if (low == high){ return false; }
	mid = (high + low) / 2;
	if (arr[mid] == value){ return true; }
	if (arr[mid] < value){ return helper(value, arr, mid+1, high); }
	else { return helper(value, arr, low, mid); }
}
```