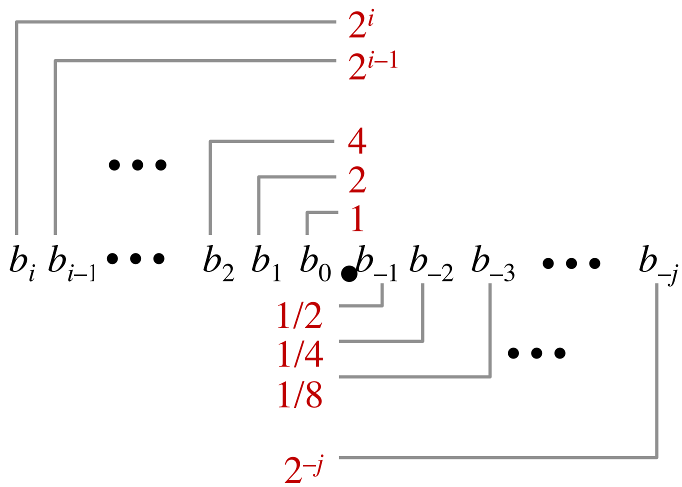
Today Topics: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 and 3/4	`101.11`$_2$
2 and 7/8	`10.111`$_2$
63/64	`0.111111`$_2$

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form `0.111111`$\ldots_2$ are just below 1.0
    - 1/2 + 1/4 + 1/8 + … + $1/2^i$ + … → 1.0
    - Use notation 1.0 – ε

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	**Representation**
1/3	`0.0101010101[01]`$\ldots_2$
1/5	`0.001100110011[0011]`$\ldots_2$
1/10	`0.0001100110011[0011]`$\ldots_2$

IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^s \, M \, 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand (mantissa) *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

- **Encoding**
  - MSB **`s`** is sign bit ***s***
  - **`frac`** field <u>encodes</u> ***M*** (but is not equal to M)
  - **`exp`** field <u>encodes</u> ***E*** (but is not equal to E)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	**8**	**23**

- **Double precision: 64 bits**

s	exp	frac
1	**11**	**52**

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	**15**	**63 or 64**

Normalized Values

- **Condition: `exp` ≠ `000…0` and `exp` ≠ `111…1`**

- **Exponent coded as *biased* value: *exp* = *E + Bias***
  - **`exp`** is an unsigned value ranging from 1 to 2^e-2
    - Allows negative values for E (= exp – Bias)
  - *Bias* = 2^{e-1} - 1, where e is number of exponent bits (bits in exp)
    - Single precision: 127 (*exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (*exp*: 1…2046, *E*: -1022…1023)

- **Significand coded with implied leading 1: *M* = `1.xxx…x`$_2$**
  - **`xxx…x`**: bits of **`frac`**
  - Minimum when **`000…0`** (*M* = 1.0)
  - Maximum when **`111…1`** (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- **Value: `Float F = 12345.0;`**
  - 12345_{10} = 11000000111001_2
    = $1.1000000111001_2 \times 2^{13}$

- **Significand**

  M = `1.`$\underline{1000000111001}_2$

  `frac`= $\underline{1000000111001}0000000000_2$

- **Exponent**

  E = 13

  Bias = 127

  `exp` = 140 = 10001100_2

- **Result:**

0	10001100	10000001110010000000000
s	exp	frac

Denormalized Values

- **Condition: `exp` = 000…0**

- **Exponent value: *E* = exp – *Bias* + 1 (instead of E = exp – Bias)**
- **Significand coded with implied leading 0: *M* = 0 . `xxx`…`x`$_2$**
  - `xxx`…`x`: bits of `frac`
- **Cases**
  - `exp` = 000…0, `frac` = 000…0
    - Represents value 0
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

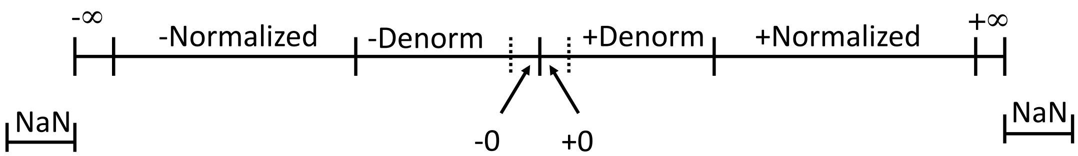
Special Values

- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −1.0/0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty * 0$

Visualization: Floating Point Encodings



Tiny Floating Point Example

s	exp	frac
1	4	3

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the **`frac`**

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

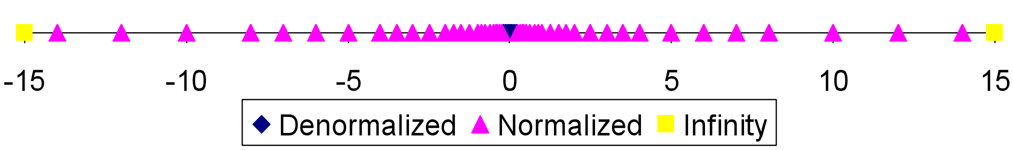
	s exp frac	*E*	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

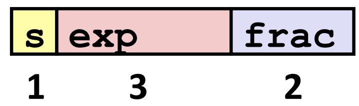
s	exp	frac
1	3	2

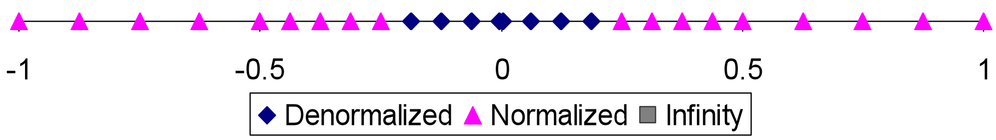
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers `{single,double}`

Description	`exp`	`frac`	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} * 2^{-\{126,1022\}}$
Single ≈ $1.4 * 10^{-45}$			
Double ≈ $4.9 * 10^{-324}$			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) * 2^{-\{126,1022\}}$
Single ≈ $1.18 * 10^{-38}$			
Double ≈ $2.2 * 10^{-308}$			
Smallest Pos. Norm.	00…01	00…00	$1.0 * 2^{-\{126,1022\}}$
Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) * 2^{\{127,1023\}}$
Single ≈ $3.4 * 10^{38}$			
Double ≈ $1.8 * 10^{308}$			

Special Properties of Encoding

- **Floating point zero (0^+) exactly the same bits as integer zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x *_f y = \text{Round}(x * y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	\$1.40	**\$1.60**	**\$1.50**	**\$2.50**	**–\$1.50**
Towards zero	\$1	\$1	\$1	\$2	–\$1
Round down (-∞)	\$1	\$1	\$1	\$2	–\$2
Round up (+∞)	\$2	\$2	\$2	\$3	–\$1
Nearest (default)	\$1	\$2	\$2	\$2	–\$2

- **What are the advantages of the modes?**

Closer Look at Round-To-Nearest

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Floating Point Multiplication

$(-1)^{s1}\ M1\ \ 2^{E1}\ \ *\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: $s1 \wedge s2$
  - Significand M: $M1 * M2$
  - Exponent E: $E1 + E2$

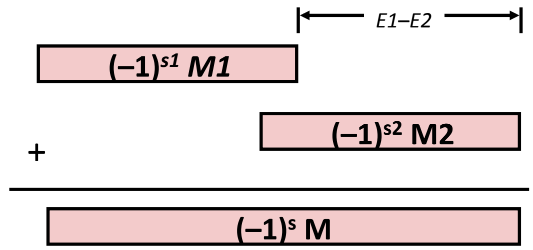
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

$(-1)^{s1}$ M1 2^{E1} + $(-1)^{s2}$ M2 2^{E2}

Assume $E1 > E2$



- **Exact Result: $(-1)^s\ M\ \ 2^E$**
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$

- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Operations

- **Not really associative or distributive due to rounding**
- **Infinities and NaNs cause issues (e.g., no additive inverse)**
- **Overflow and infinity**

Floating Point in C

- **C Guarantees Two Levels**

  `float` single precision

  `double` double precision

- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`Double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as int has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Memory Referencing Bug (Revisited)

```
double fun(int i)
{
  volatile double d[1] = {3.14};
  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
  return d[0];
}
```

```
fun(0)  ->    3.14
fun(1)  ->    3.14
fun(2)  ->    3.1399998664856
fun(3)  ->    2.00000061035156
fun(4)  ->    3.14, then segmentation fault
```

Explanation:

`Saved State`	4
`d7 … d4`	3
`d3 … d0`	2
`a[1]`	1
`a[0]`	0

Location accessed by `fun(i)`

Representing 3.14 as a Double FP Number

- **1073741824 = 0100 0000 0000 0000 0000 0000 0000 0000**
- **3.14 = 11.0010 0011 1101 0111 0000 1010 000…**
- **$(-1)^s\ M\ \ 2^E$**
  - S = 0 encoded as 0
  - M = 1.1001 0001 1110 1011 1000 0101 000…. (leading 1 left out)
  - E = 1 encoded as 1024 (with bias)

`s`	`exp (11)`	`frac (first 20 bits)`
0	**100 0000 0000**	**1001 0001 1110 1011 1000**

`frac (another 32 bits)`
0101 0000 …

Memory Referencing Bug (Revisited)

```
double fun(int i)
{
  volatile double d[1] = {3.14};
  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
  return d[0];
}
```

```
fun(0)  ->     3.14
fun(1)  ->     3.14
fun(2)  ->     3.1399998664856
fun(3)  ->     2.00000061035156
fun(4)  ->     3.14, then segmentation fault
```

`Saved State`		4	
`d7` … `d4`	0100 0000 0000 0000 0000 0000 0000 0000	3	Location accessed by `fun(i)`
`d3` … `d0`	0100 0000 0000 0000 0000 0000 0000 0000	2	
`a[1]`		1	
`a[0]`		0	

Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form $M \times 2^E$**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers