Today: Floats!

Hi! :)

Today Topics: Floating Point

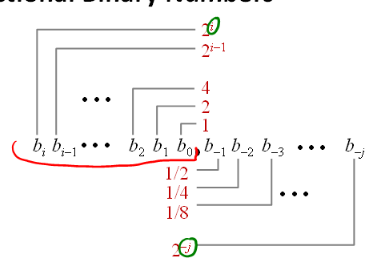
- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101?

11.5?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers: Examples

Value		Representation	
5 and 3/4	5	$\underbrace{101}.\underbrace{11}_2$	2^{-1} 2^{-2} $\frac{1}{2} + \frac{1}{4}$
2 and 7/8		$10.\underbrace{111}_2$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$
63/64		0.111111_2	

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	Representation
→ 1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

Fixed Point Representation

32/64

- float → 32 bits; double → 64 bits
- We might try representing fractional binary numbers by picking a fixed place for an implied binary point
  - "fixed point binary numbers"
- Let's do that, using 8 bit floating point numbers as an example
  - #1: the binary point is between bits 2 and 3
    $b_7\ b_6\ b_5 b_4\ b_3\ [.]\ b_2\ b_1\ b_0$
  - #2: the binary point is between bits 4 and 5
    $b_7\ b_6\ b_5\ [.]\ b_4\ b_3\ b_2\ b_1\ b_0$
  - The position of the binary point affects the range and precision
    - range: difference between the largest and smallest representable numbers
    - precision: smallest possible difference between any two numbers

Fixed Point Pros and Cons

- Pros
  - It's simple. The same hardware that does integer arithmetic can do fixed point arithmetic
    - In fact, the programmer can use ints with an implicit fixed point
      - E.g., int balance; // number of pennies in the account
    - ints are just fixed point numbers with the binary point to the right of b_0
- Cons
  - There is no good way to pick where the fixed point should be
    - Sometimes you need range, sometimes you need precision. The more you have of one, the less of the other

What else could we do?

precision

range

1.2

.20

IEEE Floating Point

- Fixing fixed point: analogous to scientific notation
  - Not 12000000 but 1.2 x 10^7; not 0.0000012 but 1.2 x 10^-6
- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- Numerical Form: can be negative < 1.0
  $(-1)^s\ M\ 2^E$
  - Sign bit *s* determines whether number is negative or positive
  - Significand (mantissa) *M* normally a fractional value in range [1.0,2.0).
  - Exponent *E* weights value by power of two
- Encoding
  - MSB s is sign bit *s*
  - `frac` field encodes *M* (but is not equal to M)
  - `exp` field encodes *E* (but is not equal to E)

s	exp	frac

- +

Precisions

- Single precision: 32 bits

s	exp	frac
1	8	23

- Double precision: 64 bits

s	exp	frac
1	11	52

- Extended precision: 80 bits (Intel only)

s	exp	frac
1	15	63 or 64

Normalization and Special Values

- "Normalized" means mantissa has form 1.xxxxx
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, don't bother to store it
- How do we do 0? How about 1.0/0.0?

Normalization and Special Values

- "Normalized" means mantissa has form 1.xxxxx
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, don't bother to store it
- Special values:
  - The float value 00...0 represents zero
  - If the exp == 11...1 and the mantissa == 00...0, it represents ∞
  - E.g., 10.0 / 0.0 → ∞
- If the exp == 11...1 and the mantissa != 00...0, it represents NaN
  - "Not a Number"
  - Results from operations with undefined result
    - E.g., 0 * ∞

How do we do operations?

- Is representation exact? no
- How are the operations carried out?

Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x *f y = Round(x * y)`
- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Floating Point Multiplication

$(-1)^{s1}\ M1\ 2^{E1}\ *\ (-1)^{s2}\ M2\ 2^{E2}$

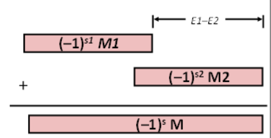
- Exact Result: $(-1)^s\ M\ 2^E$
  - Sign *s*: *s1 ^ s2*
  - Significand *M*: *M1 * M2*
  - Exponent *E*: *E1 + E2*
- Fixing
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision
- Implementation
  - What is hardest?

Floating Point Addition

$(-1)^{s1}\ M1\ 2^{E1}\ +\ (-1)^{s2}\ M2\ 2^{E2}$

Assume *E1 > E2*

- Exact Result: $(-1)^s\ M\ 2^E$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*



- Fixing
  - If *M* ≥ 2, shift *M* right, increment *E*
  - if *M* < 1, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision

Hmm... if we round at every operation...

(a+b)+c

Mathematical Properties of FP Operations

- Not really associative or distributive due to rounding
- Infinities and NaNs cause issues
- Overflow and infinity

Floating Point in C

- C Guarantees Two Levels
  - `float` single precision 32
  - `double` double precision 64

double d;
float f = 1.0;
d = (double) f;
int i = (int) f;

- Conversions/Casting
  - Casting between `int`, `float`, and `double` changes bit representation
  - `Double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, why?
  - `int` → `float`
    - Will round according to rounding mode

Memory Referencing Bug (Revisited)

```
double fun(int i)
{
  volatile double d[1] = {3.14};
  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
  return d[0];
}
```

```
fun(0)  ->   3.14
fun(1)  ->   3.14
fun(2)  ->   3.1399998664856
fun(3)  ->   2.00000061035156
fun(4)  ->   3.14, then segmentation fault
```

Explanation:

~~Saved State~~	4	Location accessed by `fun(i)`
d7 … d4	3	
~~d3 … d0~~	2	
a[1]	1	
a[0]	0	

d7 d0 | s | exp | M |
11 52

Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {

  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for ( i=0; i<10; i++ ) {
    f2 += 1.0/10.0;
  }
      f2 = 1.0000.... ;
  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
  printf("f2 = %10.8f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf ("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```


Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {

  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for ( i=0; i<10; i++ ) {
    f2 += 1.0/10.0;
  }
           0.1
  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
  printf("f2 = %10.8f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf ("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```

Summary

- As with integers, floats suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow, just like ints
  - Some “simple fractions” have no exact representation
    - E.g., 0.1
  - Can also lose precision, unlike ints
    - “Every operation gets a slightly wrong result”
- Mathematically equivalent ways of writing an expression may compute differing results
- NEVER test floating point values for equality!