Floating Point II

CSE 351 Autumn 2022

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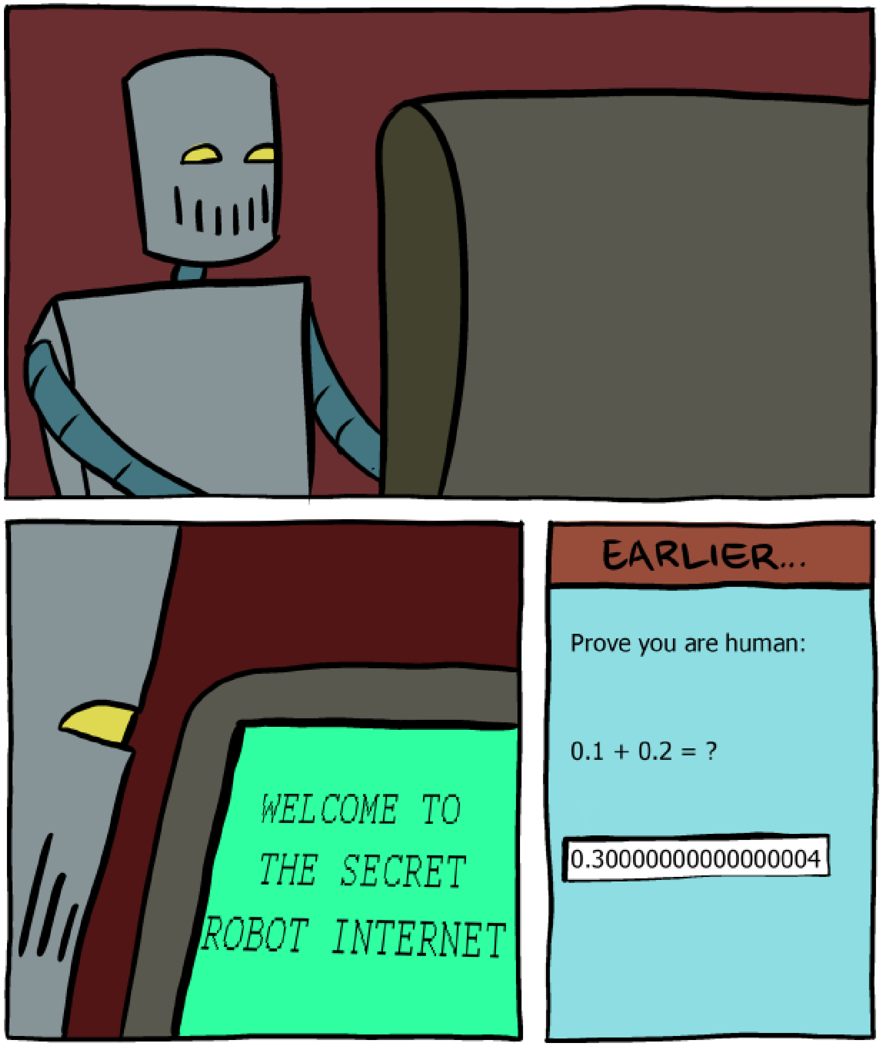

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Relevant Course Information

- hw6 due Friday, hw7 due Monday
- Lab 1a: last chance to submit is tonight @ 11:59 pm
  - One submission per partnership
  - Make sure you check the Gradescope autograder output!
  - Grades hopefully released by end of Sunday (10/16)
- Lab 1b due Monday (10/17)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`
- Section tomorrow on Integers and Floating Point

Getting Help with 351

- Lecture recordings, readings, inked slides, section presentation recordings, worksheet solutions
- Form a study group!
  - Good for everything but labs, which should be done in pairs
  - Communicate regularly, use the class terminology, ask and answer each others' questions, show up to OH together
- Attend office hours
  - Can also chat with other students– help each other learn!
- Post on Ed Discussion
- Request a 1-on-1 meeting
  - Available on a limited basis for special circumstances

Reading Review

- Terminology:
  - Special cases
    - Denormalized numbers
    - $\pm\infty$
    - Not-a-Number (NaN)
  - Limits of representation
    - Overflow
    - Underflow
    - Rounding

- Questions from the Reading?

Review Questions

- What is the value of the following floats?
  - 0x00000000 ⇒ S=0, E=0, M=0 ⇒ +0
  - 0xFF800000 ⇒ S=1, E= all 1's, M=0 ⇒ −∞

  0b 1 | 1111 1111 | 1000 0…0 (S, E, M)

- For the following code, what is the smallest value of n that will encounter a limit of representation?

```
float f = 1.0;  // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024;  // 1024 = 2^10
printf("f = %f\n", f);
```

n	f
1	2^{10}
2	2^{20}
3	2^{30}
4	2^{40}
⋮	⋮

always M = 0

E_{max} = 0xFE, Exp_{max} = 254 − 127 = 127

for n = 13, we hit 2^{130}, which causes overflow

Floating Point Encoding Summary (Review)

E	M	Interpretation
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

smallest E (all 0's) — rows 1–2
everything else — row 3
largest E (all 1's) — rows 4–5

Special Cases

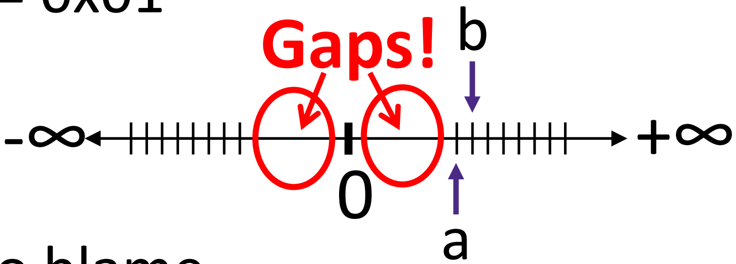
- But wait… what happened to zero?
  - *Special case:* E and M all zeros = 0
  - Two zeros! But at least 0x00000000 = 0 like integers
    0x80000000 = -0
- E = 0xFF, M = 0: ± ∞
  - *e.g.*, division by 0
  - Still work in comparisons!

all ones

- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.*, square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging (tells you cause of NaN)

New Representation Limits (Review)

- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$
- New numbers closest to 0:
  - E = 0x00 taken; next smallest is E = 0x01
  - $a = 1.0\ldots00_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers (Review)

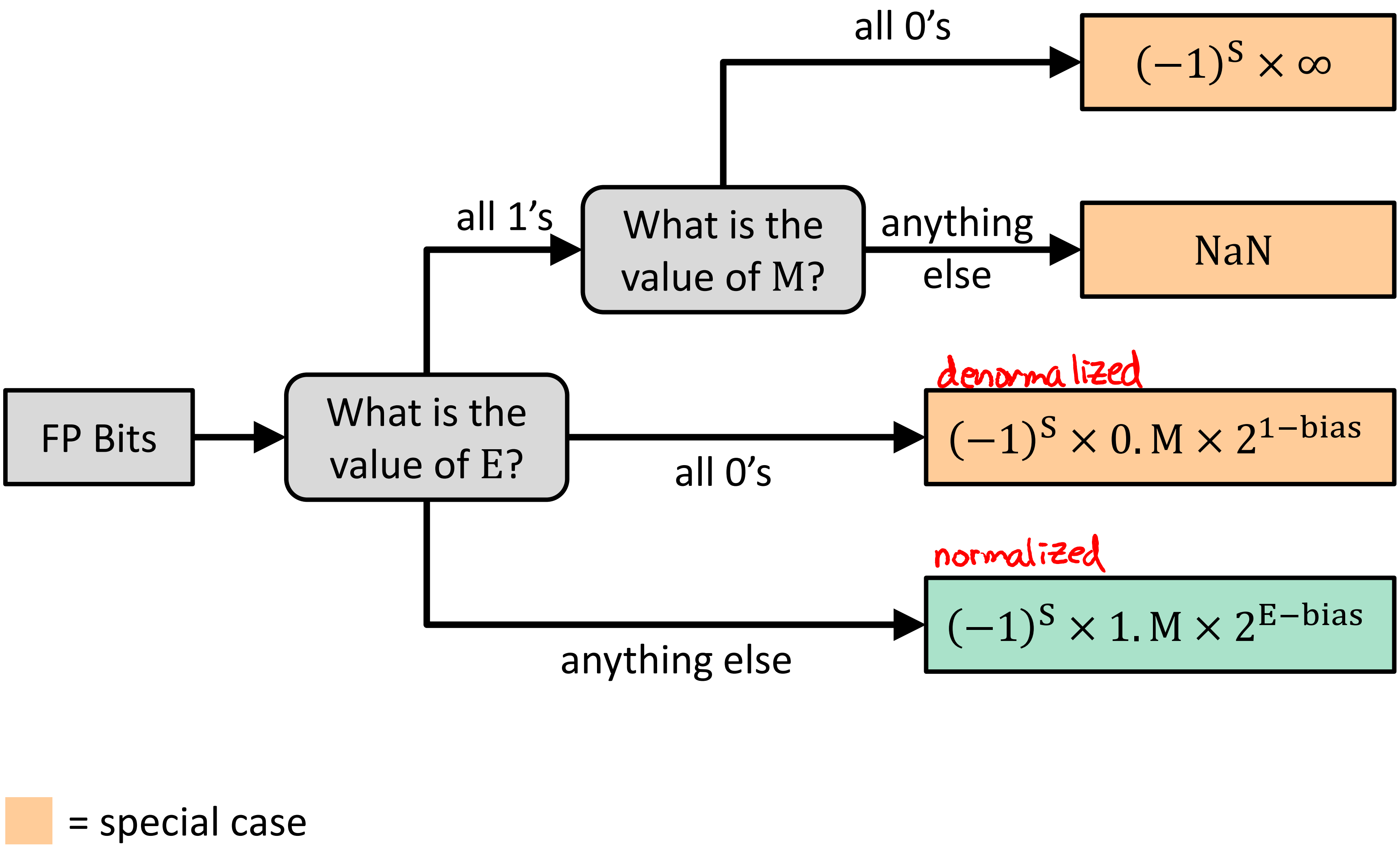
This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\, 1.0\ldots0_{two}\times 2^{-126} = \pm\, 2^{-126}$
  - Smallest denorm: $\pm\, 0.0\ldots01_{two}\times 2^{-126} = \pm\, 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point Decoding Flow Chart



Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and limitations**
- **Floating-point in C**

- There are many more details that we won't cover
  - It's a 58-page standard…

Tiny Floating Point Representation

- We will use the following **8-bit** floating point representation to illustrate some key points:

S	E	M
1	4	3

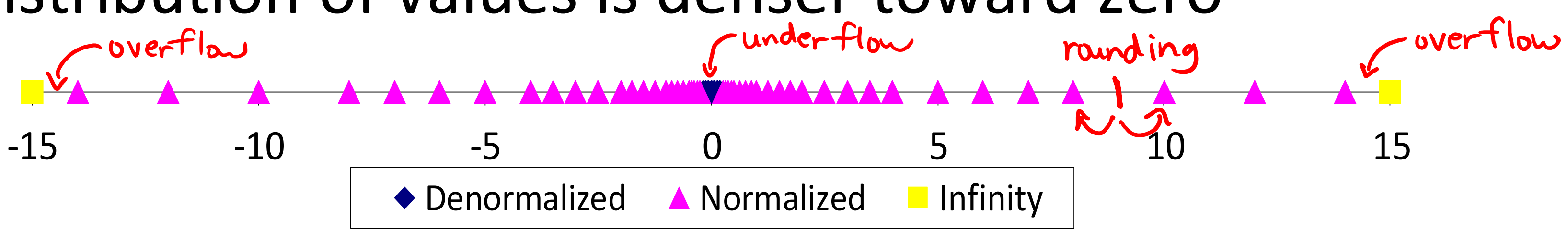
- Assume that it has the same properties as IEEE floating point:
  - bias = $2^{w-1}-1 = 2^{4-1}-1 = 7$
  - encoding of −0 = 0b 1 0000 000
  - encoding of +∞ = 0b 0 1111 000
  - encoding of the largest (+) normalized # = 0b 0 1110 111 → $1.111_2 \times 2^{14-7}$
  - encoding of the smallest (+) normalized # = 0b 0 0001 000 → $1.0_2 \times 2^{1-7}$

Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the next largest representable number?

  if M=0b0...00, then $2^{Exp} \times 1.0$
  if M=0b0...01, then $2^{Exp} \times (1+2^{-23})$
  diff = 2^{Exp-23}

  - What is this "step" when Exp = 0? 2^{-23}
  - What is this "step" when Exp = 100? 2^{77}
- Distribution of values is denser toward zero



Floating Point Operations: Basic Idea

Value = $(-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$

S	E	M

- `x +f y = Round(x + y)`
- `x *f y = Round(x * y)`

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

Mathematical Properties of FP Operations

- Overflow yields ±∞ and underflow yields 0
- Floats with value ±∞ and NaN can be used in operations
  - Result usually still ±∞ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`
    - 10^{100} ... 0
    - **0** ... **3.14**
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    - **30.000000000000003553** ... **30**
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)

S	E	M
1	4	3

- In our tiny example:
  - Man = 1.001/01 rounded to M = 0b001
  - Man = 1.001/11 rounded to M = 0b010
  - Man = 1.001/10 rounded to M = 0b010
  - Man = 1.000/10 rounded to M = 0b000

Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants
- `#include <float.h>` for additional constants
- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

instead use abs(f1 - f2) < 2^{-20}

↑ some arbitrary threshold

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes the bit representation** (tries to preserve the value)
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s are representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to TMin (even if the value is a very big positive)

Casting Example

- We execute the following code in C. How are `i` and `f` represented in hex?

```
int i = 384;   // 2^8 + 2^7
float f = (float) i;
```

= 0b 1 1000 0000
= $1.1_2 \times 2^8$
S = 0
E = 8 + 127 = 135
= 0b1000 0111
M = 0b 10...0
0b 0 1000 0111 100...0

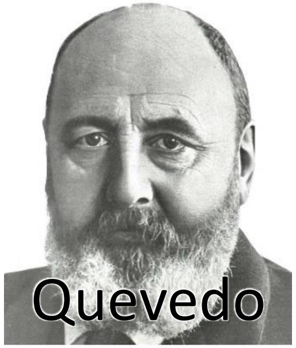
i stored as 0x 00 00 01 80
f stored as 0x 43 C0 00 00

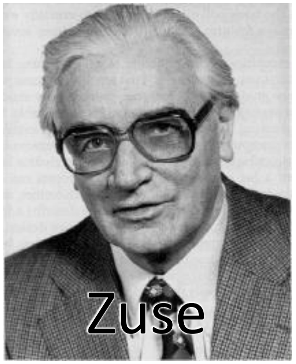
Discussion Questions

- How do you feel about floating point?
  - Do you feel like the limitations are acceptable?
  - Does this affect the way you'll think about non-integer arithmetic in the future?
  - Are there any changes or different encoding schemes that you think would be an improvement?

More on Floating Point History

- Early days
  - First design with floating-point arithmetic in 1914 by Leonardo Torres y Quevedo
  - Implementations started in 1940 by Konrad Zuse, but with differing field lengths (usually not summing to 32 bits) and different subsets of the special cases
- IEEE 754 standard created in 1985
  - Primary architect was William Kahan, who won a Turing Award for this work
  - Standardized bit encoding, well-defined behavior for *all* arithmetic operations





Floating Point in the "Wild"

- 3 formats from IEEE 754 standard widely used in computer hardware and languages
  - In C, called `float`, `double`, `long double`
- Common applications:
  - 3D graphics: textures, rendering, rotation, translation
  - "Big Data": scientific computing at scale, machine learning
- Non-standard formats in domain-specific areas:
  - **Bfloat16:** training ML models; range more valuable than precision
  - **TensorFloat-32:** Nvidia-specific hardware for Tensor Core GPUs

Type	S bits	E bits	M bits	Total bits
Half-precision	1	5	10	16
Bfloat16	1	8	7	16
TensorFloat-32	1	8	10	19
Single-precision	1	8	23	32

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.*, 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown “smart” warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

E	M	Meaning
0b0…0	anything	± denorm num (including 0)
anything else	anything	± norm num
0b1…1	0	± ∞
0b1…1	non-zero	NaN

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits