# Floating Point
## CSE 351 Spring 2024

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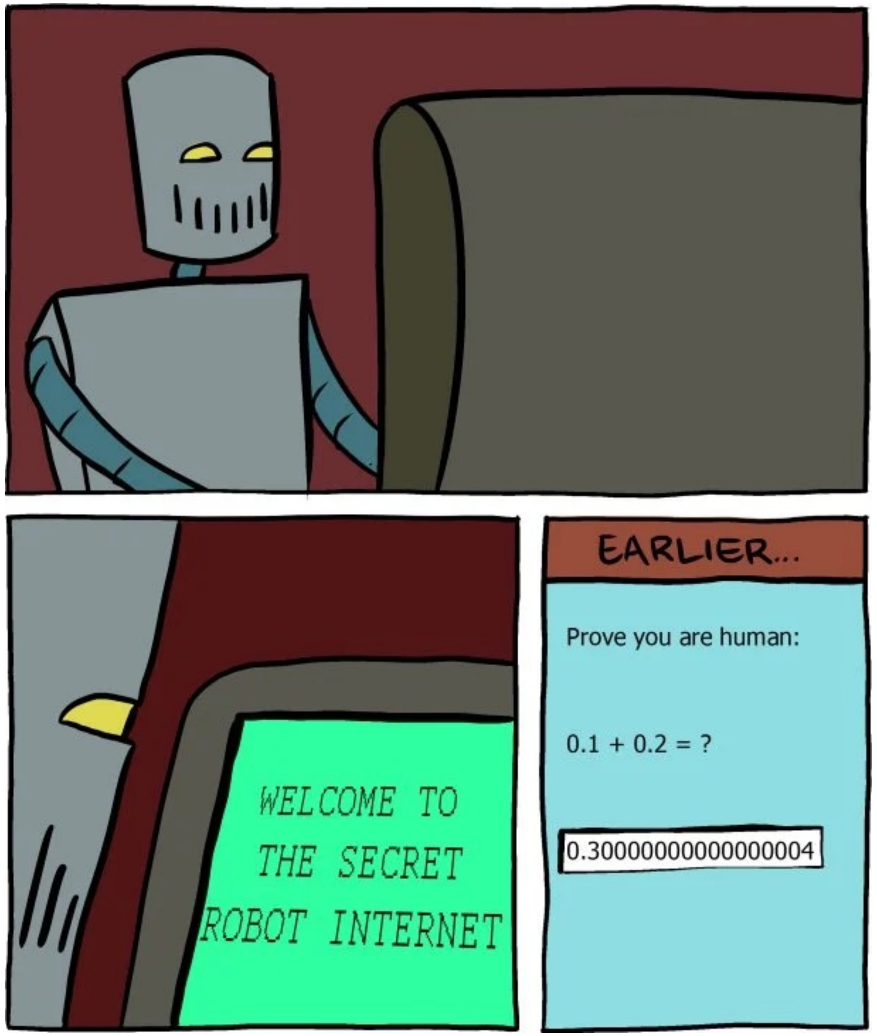


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# Announcements, Reminders

- HW4 due tonight, HW5 due Monday (8 Apr)
  - Getting ahead a bit: no RD due for Friday due to combined RD9/10!
- Lab 1a due Monday (8 Apr)
  - Submit `pointer.c` and `lab1Asynthesis.txt` on Gradescope by deadline!
    - Make sure there are no lingering `printf` statements in your code!
  - Can use (up to two) late day tokens to submit up until Wednesday 10 Apr at 11:59 PM
  - If you are submitting with a partner, ensure that <u>you add them to the submission</u>
- Lab 1b due Monday (15 Apr)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

# Exams

- Midterm and final exams will be taken on Gradescope
- Open for 48 hours and 72 hours respectively, no time limit
  - Designed to take 1-3 hours
  - Midterm open May 6$^{th}$ at 00:00, due May 7$^{th}$ at 23:59
  - Final open June 3$^{rd}$ at 00:00, due June 5$^{th}$ at 23:59
- Open book, open notes, open (.*)
  - But <u>not</u> group work—taken individually
  - High-level discussion with classmates OK, but you must write answers on your own (like labs, but without a partner)
- Mixture of problem-solving, design, and personal reflection questions (short answer & open ended)

# Lab 1b Aside: C Macros

from LC4

- C macros basics: e.g. #define SUIT_MASK 0x30
  - Basic syntax is of the form: `#define NAME expression`
  - Allows you to use `NAME` instead of `expression` in code
    - Does naïve copy and replace before compilation – everywhere the characters `NAME` appear in the code, the characters `expression` will now appear instead
    - **Not** the same as a Java constant, but used in a similar way
  - Useful to help with readability/factoring in code

- You'll use C macros in Lab 1b for defining bit masks
  - See Lab 1b starter code and LC4 slides (card operations) for examples

# Reading Review

- Terminology:
  - normalized scientific binary notation
  - trailing zeros
  - sign, mantissa, exponent ↔ bit fields S, M, and E
  - `float`, `double`
  - biased notation (exponent), implicit leading one (mantissa)
  - rounding errors

# Review Questions

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

$\overbrace{11}^{8+2+1}.\overbrace{375}^{0.25+0.125}$

- Convert $11.375_{10}$ to normalized binary scientific notation

$1\ 0\ 1\ 1\ .\ 0\ 1\ 1 \xrightarrow{\text{normalize}} 1.011011 \times 2^3$

- What is the value (in decimal) encoded by the following floating-point number?

S E M

`0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000`

Sign = +   exponent = 128 − 127 = 1   Mantissa = 1.11 (implicit 1)

$\Longrightarrow +1.11 \times 2^1$

exponent (E)   Sign (S)   mantissa (M)

- exponent = E – bias, where bias = $2^{w-1}$-1 $= 2^{8-1} - 1 = 127$
- mantissa = 1.M

not normalized $= 11.1 = 3.5_{10}$

# Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (e.g., 3.14159)
  - Very large numbers (e.g., $6.02 \times 10^{23}$)
  - Very small numbers (e.g., $6.626 \times 10^{-34}$)
  - Special numbers (e.g., ∞, NaN)

**Floating Point**

**Cram it all into one encoding?!**

# Floating Point Topics

- **Fractional binary numbers**
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

# Representation of Fractions

- **Binary Point**, like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:

$$\underset{2^1\ 2^0}{xx}.\underset{2^{-1}\ 2^{-2}\ 2^{-3}\ 2^{-4}}{yyyy}$$

What if this binary point wasn't fixed?

We could represent a larger range of values! Let's make it float!

- Example:

$$10.1010_2 = 1\times 2^1 + 1\times 2^{-1} + 1\times 2^{-3} = 2.625_{10}$$

# Binary Scientific Notation (Review)

$$\overbrace{1.01}^{\text{mantissa}}{}_2 \times 2^{-1}$$

exponent

binary point

radix (base)

- **Normalized form**: exactly one digit (non-zero) to left of binary point
- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

# IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**.
  - Who won?

    Scientists mostly won out:
    - Nice standards for rounding, overflow, underflow, but... complex for hardware
    - **Float operations can be an order of magnitude slower than integer ops ➔ so slow, it's used as a performance gauge! (e.g. FLOPS/s)**

# Floating Point Encoding (Review)

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**

| 31 | 30 … 23 | 22 … 0 |
|---|---|---|
| **S** | **E** | **M** |
| **1 bit** | **8 bits** | **23 bits** |

# The Exponent Field (Review)

- Use biased notation
  - Read exponent as unsigned, but with a **bias** of $2^{w-1}-1$ (bias = 127, for E of 8 bits)
  - Representable exponents roughly ½ positive and ½ negative
  - Exp = E – bias ↔ E = Exp + bias
    - Exponent value of 0 (Exp = 0) is thus represented as E = 0b 0111 1111

- Why biased?
  - Makes floating point arithmetic easier—somewhat compatible with two's complement hardware. Very hand wave-y, I know.
  - Now it's a sign-and-magnitude representation!

# The Mantissa (Fraction) Field (Review)

| 31 | 30 ... 23 | 22 ... 0 |
|---|---|---|
| S | E | M |
| 1 bit | 8 bits | 23 bits |

$$(-1)^S \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the **implicit leading 1** in front of M bit vector (Normalized form)
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000
    Read as $1.1_2 = 1.5_{10}$, not $0.1_2 = 0.5_{10}$, because of implicit leading 1
  - A "free" extra bit of precision!
- Mantissa "limits"
  - Low values near M = 0b000…000 are close to $2^{\text{Exp}}$
  - High values near M = 0b111…111 are close to $2^{\text{Exp}+1}$

$1.0000\ldots \times 2^{Exp}$

$1.11111\ldots \times 2^{Exp}$

Difference b/w $2^{Exp+1}$ & $1.1111\ldots \times 2^{Exp}$ is small:

$2^{Exp+1} - \underbrace{2^{Exp-23}}_{\text{Small but finite!}}$

# Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of $\mathrm{M}$ to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{\mathrm{E} - \text{bias}}$.
  3. Multiply the sign $(-1)^{S}$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as $S$ (0/1).
  4. Add the bias to exponent and encode $\mathrm{E}$ as unsigned.
  5. The first bits after the leading 1 that fit are encoded into $\mathrm{M}$.

# Example & Practice Question

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

- Convert the decimal number **-11.375** into floating point representation

$S = 1$ (Since -11.375 is negative)

$-1011.011 \xrightarrow{\text{normalize}} -1.\underbrace{011011}_{\text{Mantissa}} \times 2^{3}$ ← Exponent

$E = 3_{10} + 127_{10} = 130_{10} \rightarrow$ 0b10000010

confused? See Slide 13!

$M = 011011$

∴ Floating Point ⟹ [0b 1100 | 0001 | 0011 | 0110 | 0000 | 0... ]
(C 1 3 6)

In hex: 0xC136

Exponent = E – bias ↔ E = Exponent + bias

Mantissa = 1.M

# Precision and Accuracy

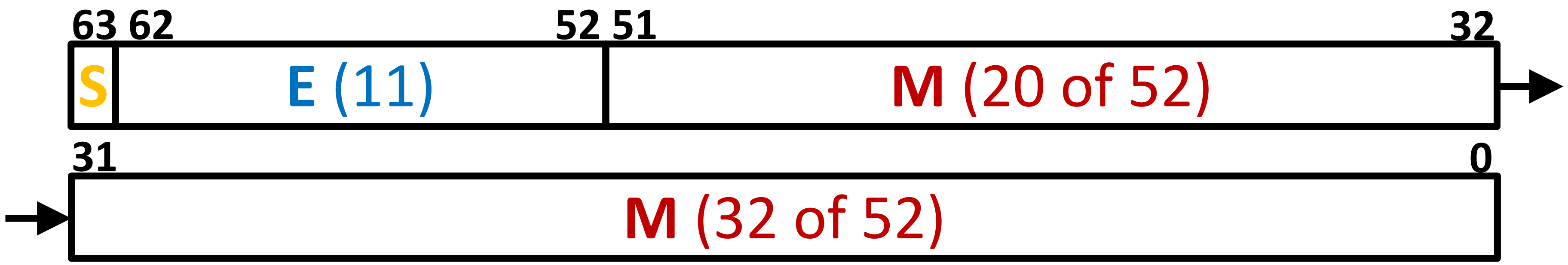
- **Precision** is a count of the number of bits in a computer word used to represent a value, i.e. **capacity** for accuracy
- **Accuracy** is a measure of the difference between the **actual value of a number** and its computer representation

**High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.**

- **Example:** `float pi = 3.14;`
  - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

# Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits




  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

# Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- **Floating-point in C**

- There are many more details that we won't cover
  - It's a 58-page standard…

# Special Cases & Encodings

But wait, how to represent zero & other fun stuff...?

- **Case 1:** E and M all zeros → **0**
  - Wait, what about the S bit? Two zeros!

    But at least 0x00000000 = 0 like integers

- **Case 2:** E = 0xFF, M = 0 → **± ∞**
  - *e.g.*, division by 0
  - Still work in comparisons!

- **Case 3:** E = 0xFF, M ≠ 0 → Not a Number (**NaN**)
  - *e.g.*, square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging

We need a specific encoding for 0 because otherwise:

$1.0 \times 2^0 = \boxed{1_{10}}$

Mantissa of 0...

exponent of 0...

Implicit 1 ruins it!

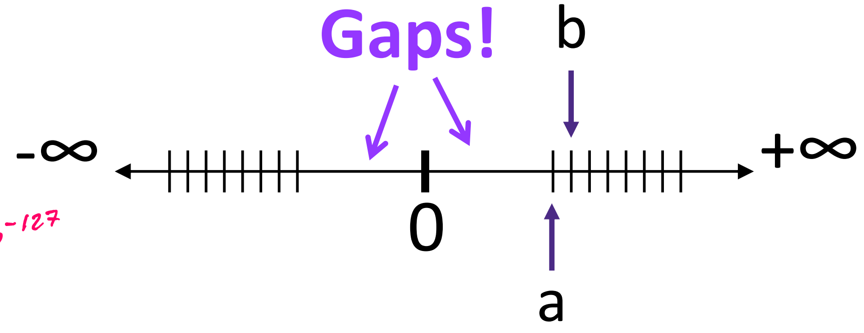
# New Representation Limits due to Special Cases

- What's now the largest value (besides ∞)?
  - E = 0xFF has now been taken by **Case 2** & **Case 3**!
  - E = 0xFE is now largest: $\boxed{1.1\ldots1_2 \times 2^{127}} = \boxed{2^{128} - 2^{104}}$

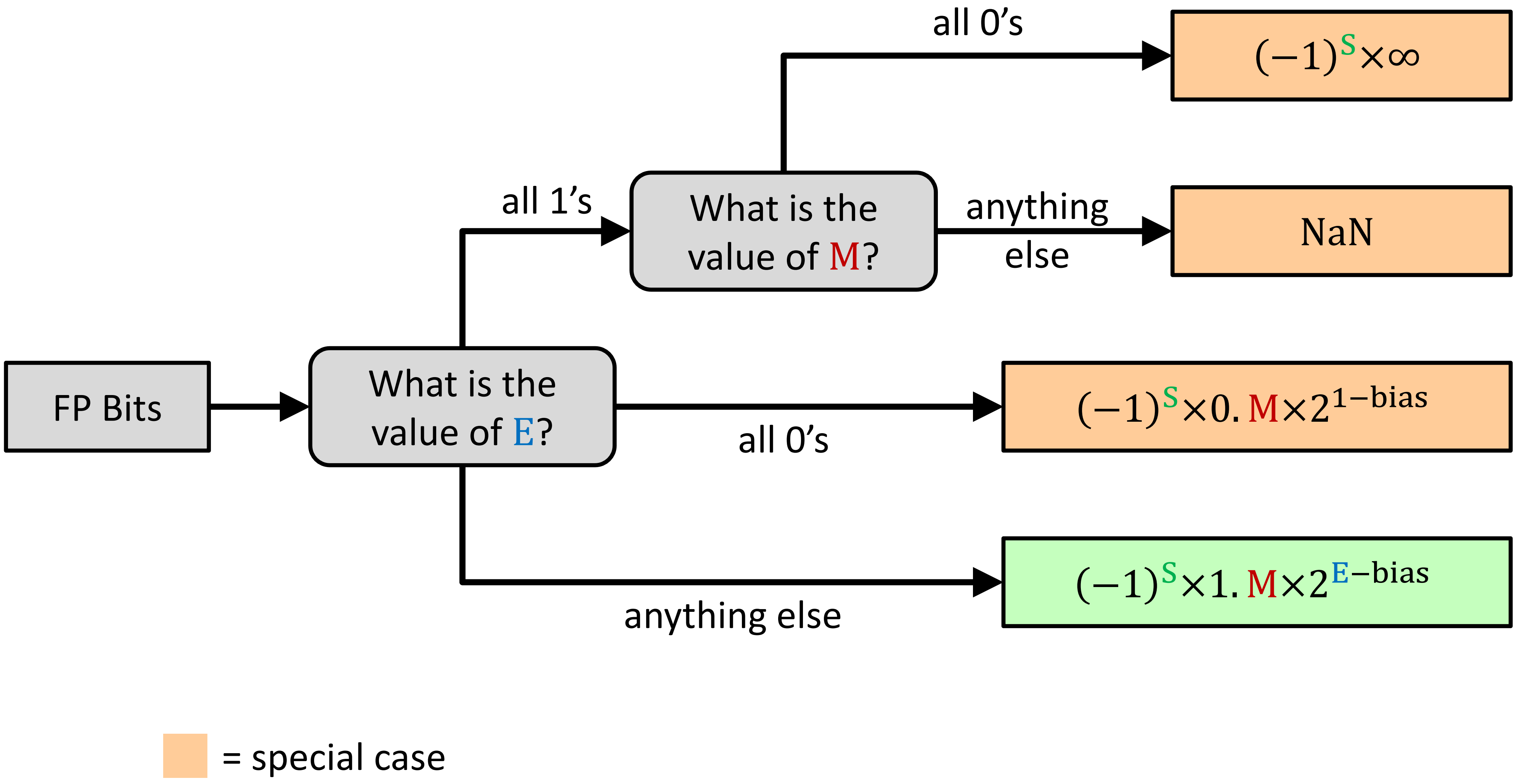
    Used to be able to be $2^{128}$

- What are now the numbers closest to 0? (i.e. M = 0)
  - E = 0x00 taken by **Case 1**; so next smallest is E = 0x01

    $2^{-126}$, used to be able to be $2^{-127}$
  - $a = 1.0\ldots00_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit leading 1 are to blame
  - Leads to another Special case: E = 0, M ≠ 0 are **denormalized numbers**
    - Mantissa has implicit leading 0 instead of implicit leading 1
    - Store much smaller numbers



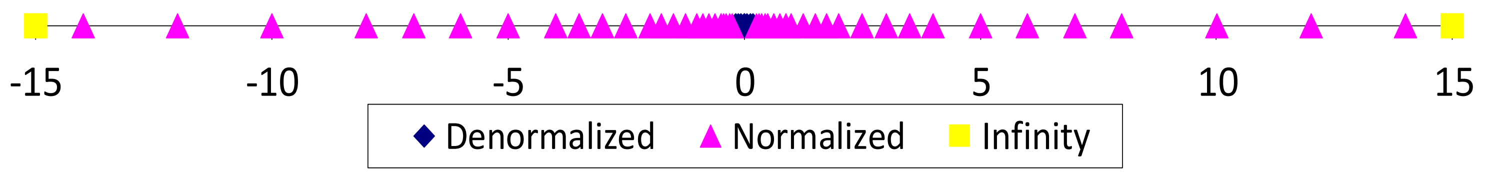
# Floating Point Decoding Flow Chart



# Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the next largest representable number?
  - What is this "step" when Exp = 0? $\mathbf{2^{-23}}$
  - What is this "step" when Exp = 100? $\mathbf{2^{77}}$
- Distribution of values is denser toward zero:

**You can represent really large numbers, or really precise numbers, but not both!**



# Floating Point Operations: Basic Idea

Value = $(-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$

| S | E | M |
|---|---|---|

- `x +f y = Round(x + y)`
- `x *f y = Round(x * y)`

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

# Mathematical Properties of FP Operations

- Overflow yields $\pm\infty$ and underflow yields 0
- Floats with value $\pm\infty$ and NaN can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)–1e100 != 3.14+(1e100–1e100)`
    - `0` vs. `3.14`
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    - `30.000000000000003553` vs. `30`
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

# Floating Point in C

- Two common levels of precision:

| | | |
|---|---|---|
| `float` | `1.0f` | single precision (32-bit) |
| `double` | `1.0` | double precision (64-bit) |

- `#include <math.h>` to get `INFINITY` and `NAN` constants
- `#include <float.h>` for additional constants

- **Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!**

Instead do something like

$if((x-y) < \varepsilon)\ldots$

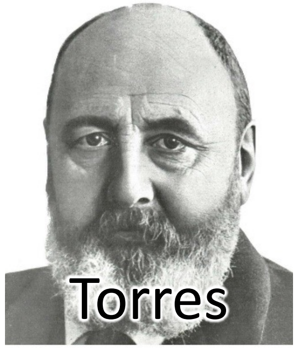
for some small epsilon value...

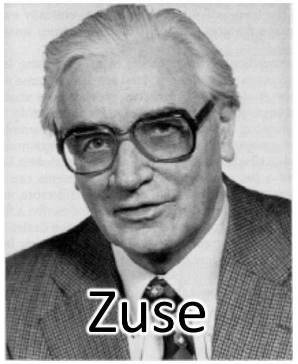
# Floating Point Conversions in C

- Casting between `int`, `float`, and `double` changes the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s are representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to TMin (even if the value is a very big positive)

# More on Floating Point History

- Early days
  - First design with floating-point arithmetic in 1914 by Leonardo Torres y Quevedo
  - Implementations started in 1940 by Konrad Zuse, but with differing field lengths (usually not summing to 32 bits) and different subsets of the special cases
- IEEE 754 standard created in **1985**
  - Primary architect was William Kahan, who won a Turing Award for this work
  - Standardized bit encoding, well-defined behavior for *all* arithmetic operations




# Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to **`TMin`** in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

# Summary

- Floating point approximates real numbers:

| 31 | 30 23 | 22 0 |
|---|---|---|
| S | E (8) | M (23) |

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1} – 1$)
    - Size of exponent field determines our *representable range*
    - Outside of representable exponents is **overflow** and **underflow**
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

# Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `ints`
    - Some "simple fractions" have no exact representation (*e.g.*, 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- **Never** test floating point values for equality!
- Careful when converting between `ints` and `floats`!

# Summary

| E | M | Meaning |
|---|---|---|
| 0x00 | 0 | ± 0 |
| 0x00 | non-zero | ± denorm num |
| 0x01 – 0xFE | anything | ± norm num |
| 0xFF | 0 | ± ∞ |
| 0xFF | non-zero | NaN |

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits

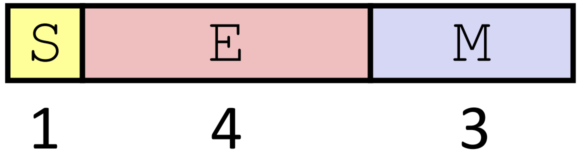
# BONUS SLIDES

Some additional information about floating point numbers. We won't test you on this, but you may find it interesting ☺

# Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)
- In our tiny example:
  - Mantissa = 1.001 01 rounded to M = 0b001
  - Mantissa = 1.001 11 rounded to M = 0b010
  - Mantissa = 1.001 10 rounded to M = 0b010
  - Mantissa = 1.000 10 rounded to M = 0b000



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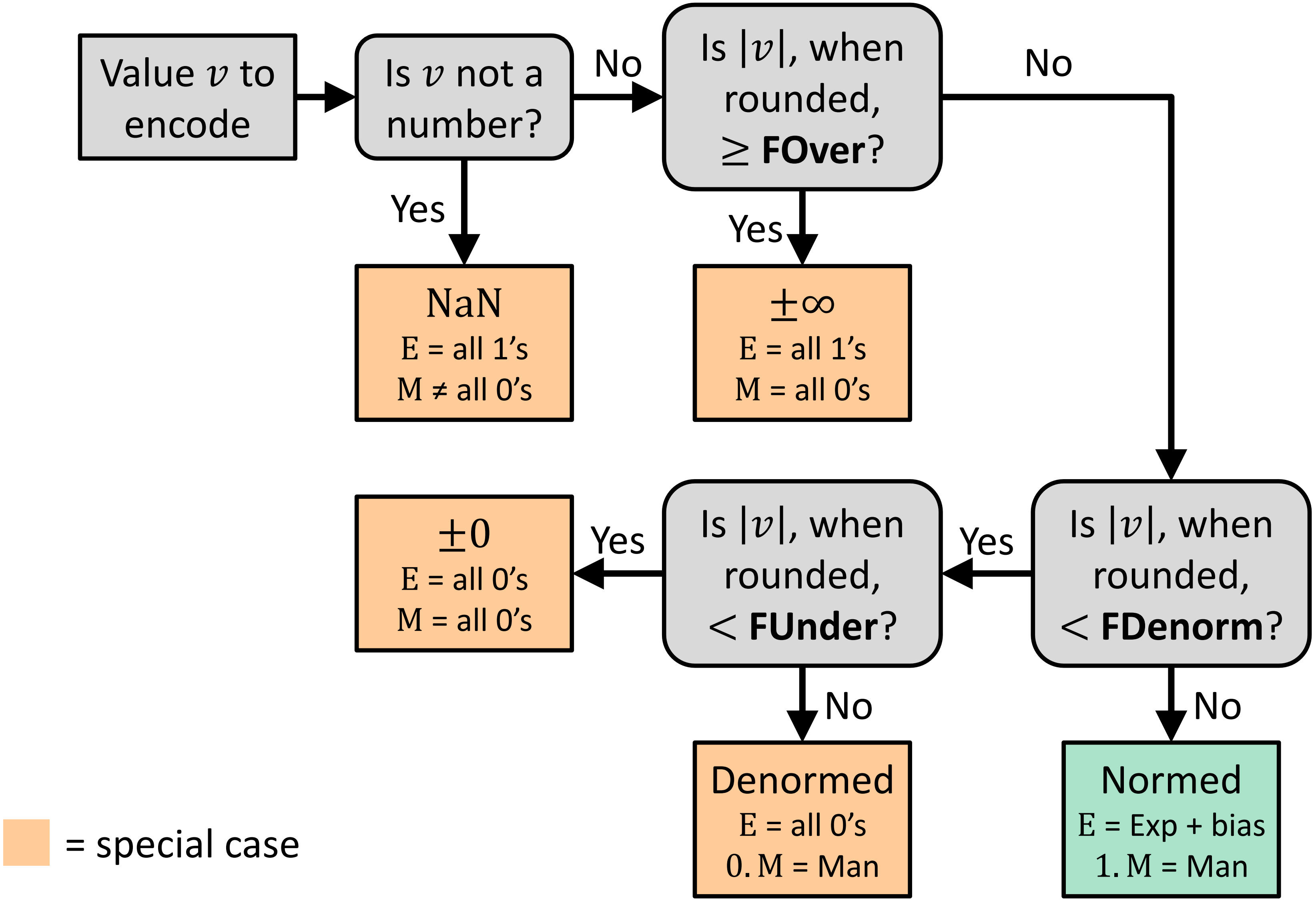
# Floating Point Encoding Flow Chart



# Limits of Interest

This is extra (non-testable) material

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = $2^8$
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = $2^{-6}$
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = $2^{-9}$
    - $m$ is the width of the mantissa field
    - This is the smallest representable denormalized number

# Denormalized Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00
- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0...0_{two} \times 2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0...01_{two} \times 2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

**So much closer to 0**

# Floating Point in the “Wild”

This is extra (non-testable) material

- 3 formats from IEEE 754 standard widely used in computer hardware and languages
  - In C, called `float`, `double`, `long double`
- Common applications:
  - 3D graphics: textures, rendering, rotation, translation
  - “Big Data”: scientific computing at scale, machine learning
- Non-standard formats in domain-specific areas:
  - **Bfloat16:** training ML models; range more valuable than precision
  - **TensorFloat-32:** Nvidia-specific hardware for Tensor Core GPUs

| Type | S bits | E bits | M bits | Total bits |
|---|---|---|---|---|
| Half-precision | 1 | 5 | 10 | 16 |
| Bfloat16 | 1 | 8 | 7 | 16 |
| TensorFloat-32 | 1 | 8 | 10 | 19 |
| Single-precision | 1 | 8 | 23 | 32 |