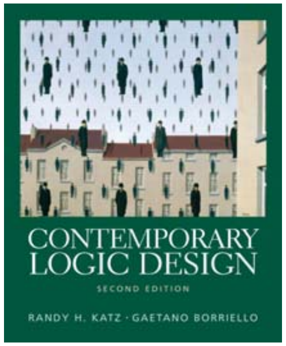
CSE 370 Spring 2006
Introduction to Digital Design

Lecture 3: Boolean Algebra



Last Lecture
- Binary and other bases
- Negative binary numbers
- Switches/CMOS

Today
- CMOS
- Basic Boolean Functions
- Boolean Algebra

Administrivia

- Hand in Homework 1: Homework 2 on the web this afternoon.

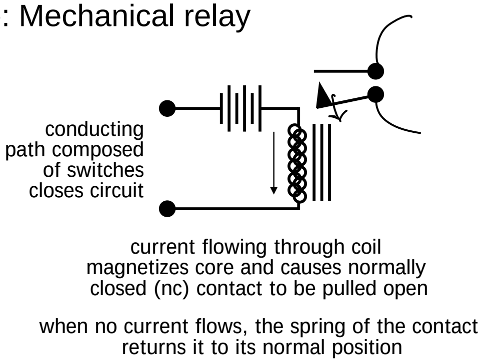
- Lab 2 is on the web, you might want to start the tutorial before you labe session.

- Office hours change:

  Adrienne Wang W 1-3pm in 003 Allen Center

Switching Networks

- Switch settings determine whether a conducting network to a light bulb
- Larger computations?
  - Use a light bulb (output) to set other switches (input)
  - Example: Mechanical relay

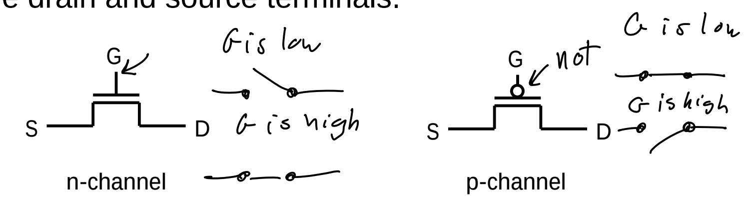


Transistor Networks

- Relays no more: slow and big
- Modern digital electronics predominately uses CMOS technology
  - MOS: metal-oxcide-semiconductor
  - C: complementary (both p and n type transistors arranged so that power is dissipated during switching.)

MOS Transistors

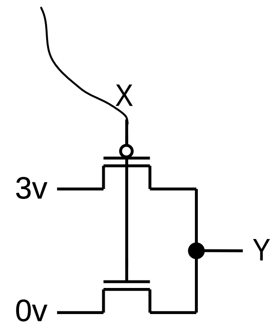
- MOS transistors have three terminals: drain, gate, and source
  - Act as switches: if the voltage on the gate terminal is (some amount) higher/lower than the source terminal then a conducting path will be established between the drain and source terminals.



n-channel
open when voltage at G is low
closes when:
voltage(G) > voltage (S) + ε

p-channel
closed when voltage at G is low
opens when:
voltage(G) < voltage (S) – ε

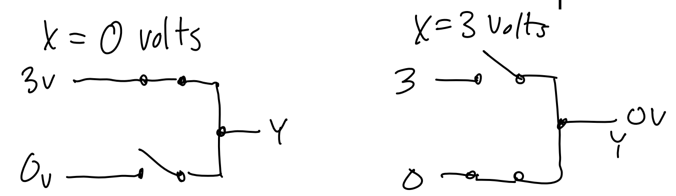
MOS Networks



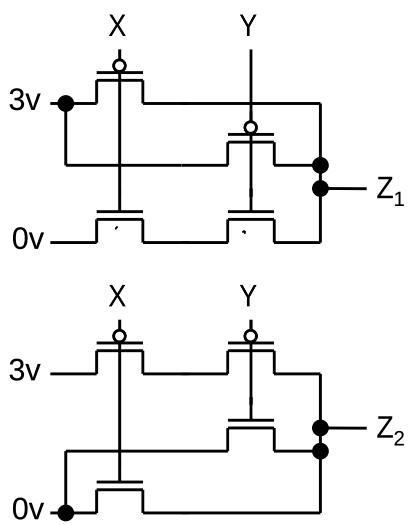
NOT X	Y
0	1
1	0

what is the relationship between x and y?

x	y
0 volts	3 Volts
3 volts	0 Volts



Two Input Networks



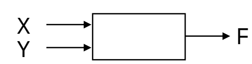
what is the relationship between x, y and z?

x	y	z1 (NAND)	z2 (NOR)
0 volts	0 volts	3V	3V
0 volts	3 volts	3V	0V
3 volts	0 volts	3V	0V
3 volts	3 volts	0V	0V
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

2 to 1 Boolean Functions

There are 16 possible two bit input one bit output

$2^4 = 16$



16 possible functions (F0–F15)

X	Y	F0															
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

0
X and Y
X
Y
X xor Y
X or Y
X nor Y
not (X or Y)
X = Y
not Y
not X
X nand Y
not (X and Y)
1

(General: k input bits, one output bit: 2^k such functions)

Costs

- 0 (F0) and 1 (F15): require 0 switches, directly connect output to low/high
- X (F3) and Y (F5): require 0 switches, output is one of inputs
- X' (F12) and Y' (F10): require 2 switches for "inverter" or NOT-gate
- X nor Y (F4) and X nand Y (F14): require 4 switches
- X or Y (F7) and X and Y (F1): require 6 switches
- X = Y (F9) and X ⊕ Y (F6): require 16 switches

NOTs, NANDs, NORs cost the least

NOT, NOR, NANDS, Oh My!

- Can we implement all logic functions from NOT, NOR, NANDs?
- Example: Implementing NOT(X NAND Y)

  is the same as implementing (X AND Y)
- In fact we can implement a NOT using a NAND or a NOR:

  NOT(X) = X NAND X   NOT(X)= Y NOR Y
- In fact NAND and NOR can be used to implement each other:

  X NAND Y=NOT(NOT(X) NOR NOT(Y))   NORs

  X NOR Y=NOT(NOT(X) NAND NOT(Y))
- To sort through the mess of what we have created we will construct a mathematical framework: Boolean Algebra

Boolean Algebra

- A set of elements B together with a two binary operations, addition, {+}, and multiplication, {•} which satisfy the *axioms*:
  - B contains at least two nonequal elements $B \times B \to B$
  - (closure) For every a,b in B

    a+b is in B   a•b is in B
  - (commutative) For every a,b in B

    a+b=b+a   a•b=b•a
  - (associative) For every a,b,c in B

    (a+b)+c=a+(b+c)   a•(b•c)=(a•b)•c
  - (identity) There exists identity elements for + and •, such that for every a in B

    a+0=a   a•1=a
  - (distributive) For every a,b,c in B

    a+(b•c)=(a+b)•(a+c)   a•(b+c)=(a•b)+(a•c)
  - (complement) For each a in B there exists an element a' in B, such that a+a'=1 and a•a'=0

A Boolean Algebra

- A Boolean Algebra:
  - the set B={0,1}
  - binary operation + = logical OR
  - binary operation • = logical AND
  - complement ' = logical NOT

$0+0=0$, $0+1=1$, $1+0=1$, $1+1=1$

$0 \cdot 0=0$, $0 \cdot 1=0$, $1 \cdot 0=0$, $1 \cdot 1=1$

$0'=1$, $1'=0$

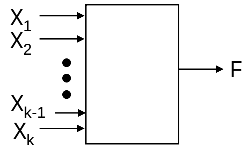
- These satisfy the above axioms
- We will often deal with variable representing an element from the set:

  Example: (X+Y)•(X+Z)

  (X OR Y) AND (X OR Z)

Boolean Functions

- Boolean Function
  - function from k input bits to one output bit



  - All such functions can be represented by a truth table

X	Y	Z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Boolean Functions and Algebra

- All Boolean Functions can be represented by an expression in Boolean Algebra using ANDs, ORs, and NOTs:

terms

2 bit

X	Y	F	$X' \cdot Y$	$X \cdot Y'$	$X' \cdot Y + X \cdot Y'$
0	0	0	0	0	0
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

$F = X' \cdot Y + X \cdot Y'$

X	Y	Z	F	
0	0	0	0	
0	0	1	1	$X' \cdot Y' \cdot Z$
0	1	0	1	$X' \cdot Y \cdot Z'$
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	1	$X \cdot Y \cdot Z$

$F = X' \cdot Y' \cdot Z + X' \cdot Y \cdot Z' + X \cdot Y \cdot Z$

Universality of NAND/NOR

- All Boolean Functions can be represented by an expression in Boolean Algebra using ANDs, ORs, and NOTs.

But we can express AND, OR, and NOT in terms of NAND:

X' = X NAND X
X AND Y = (X NAND Y)' = (X NAND Y) NAND (X NAND Y)
X OR Y = (X' NAND Y')

But we can express AND, OR, and NOT in terms of NOR:
X' = X NOR X
X OR Y = (X NOR Y)'
X AND Y = (X' NOR Y')

Duality

- All Boolean expressions have logical duals
- Any theorem that can be proved is also proved for its dual
- Replace: • with +, + with •, 0 with 1, and 1 with 0
- Leave the variables unchanged

Example: ~~X+0 = 0~~ is dual to ~~X•1=1~~
$X+0=X$ $X \cdot 1 = X$

Do not confuse Duality with de'Morgan's theorem.

$X + 0 = X$
↓ ↓ ↓ ↓ ↓
$X \cdot 1 = X$

Axioms and Theorems

1. Identity: X + 0 = X — Dual: X • 1 = X
2. Null: X + 1 = 1 — Dual: X • 0 = 0
3. Idempotent: X + X = X — Dual: X • X = X
4. Involution: (X')' = X
5. Complementarity: X + X' = 1 — Dual: X • X' = 0
6. Commutative: X + Y = Y + X — Dual: X • Y = Y • X
7. Associative: (X+Y)+Z=X+(Y+Z) — Dual: (X•Y)•Z=X•(Y•Z)
8. Distributive: X•(Y+Z)=(X•Y)+(X•Z) — Dual: X+(Y•Z)=(X+Y)•(X+Z)
9. Uniting: X•Y+X•Y'=X — Dual: (X+Y)•(X+Y')=X
10. Absorption: X + X • Y = X — Dual: X • (X + Y) = X
11. Absorption2: (X + Y') • Y = X • Y — Dual: (X • Y') + Y = X + Y
12. Factoring: (X + Y) • (X' + Z) = X • Z + X' • Y — Dual: X • Y + X' • Z = (X + Z) • (X' + Y)

Axioms and Theorems

13. Concensus: (X • Y) + (Y • Z) + (X' • Z) = X • Y + X' • Z

Dual: (X + Y) • (Y + Z) • (X' + Z) = (X + Y) • (X' + Z)

14. DeMorgan's Law: (X + Y + ...)' = X' • Y' • ...

Dual: (X • Y • ...)' = X' + Y' + ...

15. Generalized DeMorgan's Laws: f'(X1,X2,...,Xn,0,1,+,•) = f(X1',X2',...,Xn',1,0,•,+)

Notice the DeMorgan is not Duality: Duality is not a way to rewrite an expression, it is a meta-theorem.

16. Generalized Duality:

f $(X_1,X_2,...,X_n,0,1,+,\bullet) \Leftrightarrow f(X_1,X_2,...,X_n,1,0,\bullet,+)$

Proving Theorems

F=X·Y+X·Y'=X

- Example 1: Prove the uniting theorem-- X•Y+X•Y'=X

Distributive	X•Y+X•Y' = X•(Y+Y') (8)
Complementarity	= X•(1) Y+Y'=1
Identity	= X

- Example 2: Prove the absorption theorem-- X+X•Y=X

Identity	X+X•Y = (X•1)+(X•Y)
Distributive	= X•(1+Y)
Null	= X•(1)
Identity	= X

Activity

- Example 3: Prove the consensus theorem-- (XY)+(YZ)+(X'Z)= XY+X'Z

Exercise

- Example 3: Prove the consensus theorem--
  (XY)+(YZ)+(X'Z)= XY+X'Z

Complementarity XY+YZ+X'Z = XY+(X+X')YZ + X'Z
Distributive = XYZ+XY+X'YZ+X'Z

Use absorption {AB+A=A} with A=XY and B=Z

= XY+X'YZ+X'Z
Rearrange terms = XY+X'ZY+X'Z

Use absorption {AB+A=A} with A=X'Z and B=Y

XY+YZ+X'Z = XY+X'Z

Proving Theorems

- Prove by using "Perfect Induction" also called "Enumeration"
- Cumbersome for very large expressions

(X + Y)' = X' • Y'
NOR is equivalent to AND
with inputs complemented

X	Y	X'	Y'	(X + Y)'	X' • Y'
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

(X • Y)' = X' + Y'
NAND is equivalent to OR
with inputs complemented

X	Y	X'	Y'	(X • Y)'	X' + Y'
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

Logic Simplification

- Use the axioms to simplify logical expressions
  - Why? To use less hardware
- Example: A two-level logic expression

Z = A'BC + AB'C' + AB'C + ABC' + ABC
= AB'C + AB'C' + A'BC + ABC' + ABC rearrange
= AB'(C + C') + A'BC + AB(C' + C) distributive
= AB' + A'BC + AB comp.
= AB' + AB + A'BC rearrange
= A(B' + B) + A'BC distributive
= A + A'BC comp.

B'+B = 1
A·1 = A

Absorption #2D {(X •Y')+Y=X+Y} with X=BC and Y=A

Z = A + BC

Example: Full Adder

- 1-bit binary adder
  - Inputs: A, B, Carry-in
  - Outputs: Sum, Carry-out

Truth table

Cin ~~A~~	A ~~B~~	B ~~Cin~~	S	Cout
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Boolean express

S = A'B'Cin + A'BCin' + AB'Cin' + ABCin

Cout = A'BCin + AB'Cin + ABCin' + ABCin

Simplification of Carry Out

Cout = A'BCin + AB'Cin + ABCin' + ABCin
= A'BCin + AB'Cin + ABCin' + ABCin + ABCin
= A'BCin + ABCin + AB'Cin + ABCin' + ABCin
= (A'+A)BCin + AB'Cin + ABCin' + ABCin
associative = (1)BCin + AB'Cin + ABCin' + ABCin
= BCin + AB'Cin + ABCin' + ABCin + ABCin
= BCin + AB'Cin + ABCin + ABCin' + ABCin
= BCin + A(B'+B)Cin + ABCin' + ABCin
idempotent
= BCin + A(1)Cin + ABCin' + ABCin
= BCin + ACin + AB(Cin'+Cin) distribute
= BCin + ACin + AB(1)
= BCin + ACin + AB