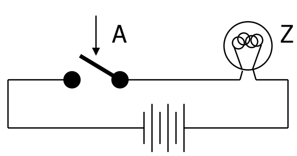
Combinational logic

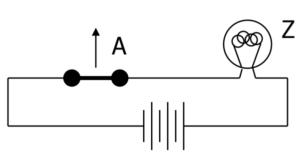
- Switches
- Basic logic and truth tables
- Logic functions
- Boolean algebra
- Proofs by re-writing and by perfect induction

Switches: basic element of physical implementations

- Implementing a simple circuit (arrow shows action if wire changes to “1”):



close switch (if A is "1" or asserted) and turn on light bulb (Z)



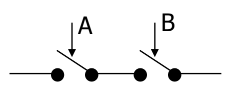
open switch (if A is "0" or unasserted) and turn off light bulb (Z)

$Z \equiv A$

Switches (cont'd)

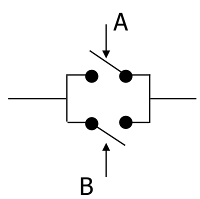
- Compose switches into more complex ones (Boolean functions):

AND



Z ≡ A and B

OR



Z ≡ A or B

Switching networks

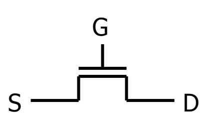
- Switch settings
  - determine whether or not a conducting path exists to light the light bulb
- To build larger computations
  - use the light bulb (output of the network)
    to set other switches (inputs to another network)

Transistor networks

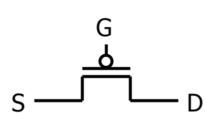
- Modern digital systems are designed in CMOS technology
  - MOS stands for Metal-Oxide on Semiconductor
  - C is for complementary because there are both normally-open and
    normally-closed switches
- MOS transistors act as voltage-controlled switches
  - similar, though easier to work with than relays.

MOS transistors

- MOS transistors have three terminals: drain, gate, and source
  - they act as switches in the following way:
    if the voltage on the gate terminal is (some amount) higher/lower than the source terminal then a conducting path will be established between the drain and source terminals

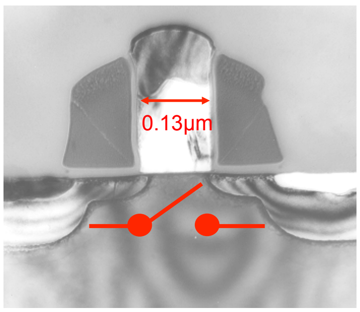


n-channel
open when voltage at G is low
closes when:
voltage(G) > voltage (S) + ε

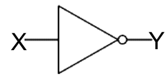


p-channel
closed when voltage at G is low
opens when:
voltage(G) < voltage (S) – ε

Most digital logic is CMOS

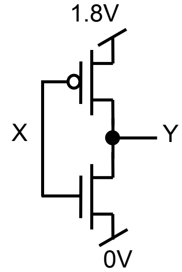


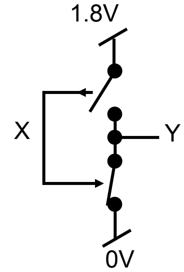
Mark Bohr Intel



0V ≡ Logic 0
1.8V ≡ Logic 1

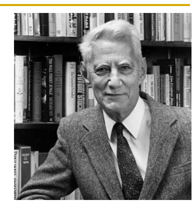
X	Y
0V	1.8V
1.8V	0V

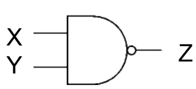




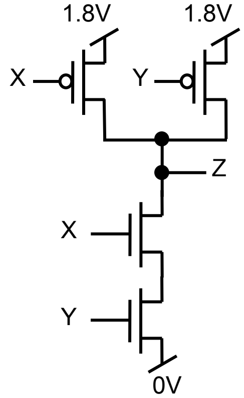
Multi-input logic gates

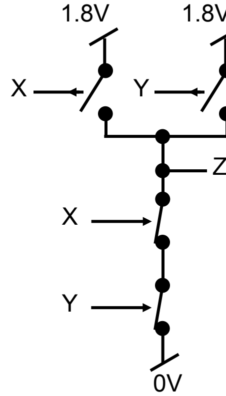
- CMOS logic gates are inverting
  - Easy to implement NAND, NOR, NOT while AND, OR, and Buffer are harder


Claude Shannon – 1938



X	Y	Z
0	0	1
0	1	1
1	0	1
1	1	0





Logic functions and Boolean algebra

- Any Boolean function can be expressed as a truth table
- Therefore it can be written as an expression in Boolean algebra using the operators: ’, +, and •

X, Y are Boolean algebra variables

X	Y	X • Y
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	X’	X’ • Y
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	0

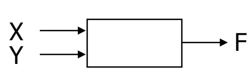
X	Y	X’	Y’	X • Y	X’ • Y’	(X • Y) + (X’ • Y’)
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

(X • Y) + (X’ • Y’) ➔ X = Y

Boolean expression that is true when the variables X and Y have the same value and false, otherwise¾¾

Possible logic functions of two variables

- There are 16 possible functions of 2 input variables:
  - in general, there are 2**(2**n) functions of n inputs



16 possible functions (F_0–F_{15})

X	Y	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		0	X and Y		X		Y	X xor Y	X or Y	X nor Y / not (X or Y)	X = Y	not Y		not X		X nand Y / not (X and Y)	1

Minimal set of functions

- Can we implement all logic functions from NOT, NOR, and NAND?
  - For example, implementing X and Y
    is the same as implementing not (X nand Y)
- In fact, we can do it with only NOR or only NAND
  - NOT is just a NAND or a NOR with both inputs tied together

X	Y	X nor Y
0	0	1
1	1	0

X	Y	X nand Y
0	0	1
1	1	0

  - and NAND and NOR are "duals",
    that is, its easy to implement one using the other

X nand Y ≡ not ((not X) nor (not Y))
X nor Y ≡ not ((not X) nand (not Y))

Boolean algebra

George Boole – 1854

- An algebraic structure consists of
  - a set of elements B
  - binary operations { + , • }
  - and a unary operation { ' }
  - such that the following axioms hold:

1. the set B contains at least two elements: a, b
2. closure: a + b is in B — a • b is in B
3. commutativity: a + b = b + a — a • b = b • a
4. associativity: a + (b + c) = (a + b) + c — a • (b • c) = (a • b) • c
5. identity: a + 0 = a — a • 1 = a
6. distributivity: a + (b • c) = (a + b) • (a + c) — a • (b + c) = (a • b) + (a • c)
7. complementarity: a + a' = 1 — a • a' = 0

Axioms and theorems of Boolean algebra

- identity
  1. X + 0 = X 1D. X • 1 = X
- null
  2. X + 1 = 1 2D. X • 0 = 0
- idempotency:
  3. X + X = X 3D. X • X = X
- involution:
  4. (X')' = X
- complementarity:
  5. X + X' = 1 5D. X • X' = 0
- commutativity:
  6. X + Y = Y + X 6D. X • Y = Y • X
- associativity:
  7. (X + Y) + Z = X + (Y + Z) 7D. (X • Y) • Z = X • (Y • Z)
- distributivity:
  8. X • (Y + Z) = (X • Y) + (X • Z) 8D. X + (Y • Z) = (X + Y) • (X + Z)

Axioms and theorems of Boolean algebra (cont'd)

- uniting:
  9. X • Y + X • Y' = X 9D. (X + Y) • (X + Y') = X
- absorption:
  10. X + X • Y = X 10D. X • (X + Y) = X
  11. (X + Y') • Y = X • Y 11D. (X • Y') + Y = X + Y
- factoring:
  12. (X + Y) • (X' + Z) = X • Z + X' • Y 12D. X • Y + X' • Z = (X + Z) • (X' + Y)
- concensus:
  13. (X • Y) + (Y • Z) + (X' • Z) = X • Y + X' • Z 13D. (X + Y) • (Y + Z) • (X' + Z) = (X + Y) • (X' + Z)
- de Morgan's:
  14. (X + Y + ...)' = X' • Y' • ... 14D. (X • Y • ...)' = X' + Y' + ...
- generalized de Morgan's:
  15. $f'(X_1,X_2,...,X_n,0,1,+,\bullet) = f(X_1',X_2',...,X_n',1,0,\bullet,+)$

Axioms and theorems of Boolean algebra (cont'd)

- Duality
  - a dual of a Boolean expression is derived by replacing • by +, + by •, 0 by 1, and 1 by 0, and leaving variables unchanged
  - any theorem that can be proven is thus also proven for its dual!
  - a meta-theorem (a theorem about theorems)
- duality:
  16. X + Y + ... ⇔ X • Y • ...
- generalized duality:
  17. $f(X_1, X_2, \ldots, X_n, 0, 1, +, \bullet) \Leftrightarrow f(X_1, X_2, \ldots, X_n, 1, 0, \bullet, +)$
- Different than deMorgan's Law
  - this is a statement about theorems
  - this is not a way to manipulate (re-write) expressions

Proving theorems (rewriting)

- Using the laws of Boolean algebra:
  - e.g., prove the theorem: X • Y + X • Y' = X

distributivity (8)	X • Y + X • Y'	=	X • (Y + Y')
complementarity (5)	X • (Y + Y')	=	X • (1)
identity (1D)	X • (1)	=	X

  - e.g., prove the theorem: X + X • Y = X

identity (1D)	X + X • Y	=	X • 1 + X • Y
distributivity (8)	X • 1 + X • Y	=	X • (1 + Y)
identity (2)	X • (1 + Y)	=	X • (1)
identity (1D)	X • (1)	=	X

Activity

- Prove consensus theorem using the laws of Boolean algebra:
  - (X • Y) + (Y • Z) + (X' • Z) = X • Y + X' • Z

identity	1. X + 0 = X	1D. X • 1 = X
null	2. X + 1 = 1	2D. X • 0 = 0
complementarity:	5. X + X' = 1	5D. X • X' = 0
commutativity:	6. X + Y = Y + X	6D. X • Y = Y • X
associativity:	7. (X + Y) + Z = X + (Y + Z)	7D. (X • Y) • Z = X • (Y • Z)
distributivity:	8. X • (Y + Z) = (X • Y) + (X • Z)	8D. X + (Y • Z) = (X + Y) • (X + Z)
factoring:	12. (X + Y) • (X' + Z) = X • Z + X' • Y	12D. X • Y + X' • Z = (X + Z) • (X' + Y)

Proving theorems (perfect induction)

- Using perfect induction (complete truth table):
  - e.g., de Morgan's:

(X + Y)′ = X′ • Y′
NOR is equivalent to AND
with inputs complemented

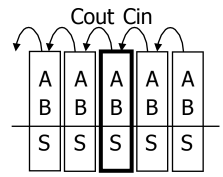
X	Y	X′	Y′	(X + Y)′	X′ • Y′
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

(X • Y)′ = X′ + Y′
NAND is equivalent to OR
with inputs complemented

X	Y	X′	Y′	(X • Y)′	X′ + Y′
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

A simple example: 1-bit binary adder

- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out



A	B	Cin	Cout	S
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

S = ____________________________________

Cout = ____________________________________