Minimization of Boolean logic

- Minimization
  - uniting theorem
  - grouping of terms in Boolean functions
- Alternate representations of Boolean functions
  - cubes
  - Karnaugh maps

Simplification of two-level combinational logic

- Finding a minimal sum of products or product of sums realization
  - exploit don't care information in the process
- Algebraic simplification
  - not an algorithmic/systematic procedure
  - how do you know when the minimum realization has been found?
- Computer-aided design tools
  - precise solutions require very long computation times, especially for functions with many inputs (> 10)
  - heuristic methods employed – "educated guesses" to reduce amount of computation and yield good if not best solutions
- Hand methods still relevant
  - to understand automatic tools and their strengths and weaknesses
  - ability to check results (on small examples)

The uniting theorem

- Key tool to simplification: A (B’ + B) = A
- Essence of simplification of two-level logic
  - find two element subsets of the ON-set where only one variable changes its value – this single varying variable can be eliminated and a single product term used to represent both elements

$$F = A'B'+AB' = (A'+A)B' = B'$$

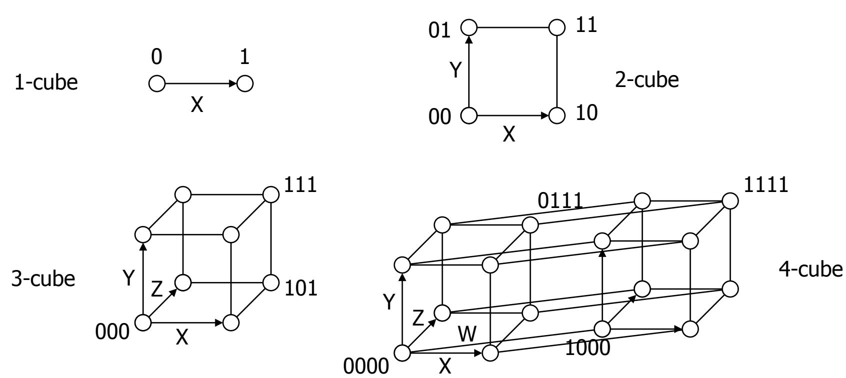
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

B has the same value in both on-set rows
– B remains, actually B’ because B is 0 in both cases

A has a different value in the two rows
– A is eliminated

Boolean cubes

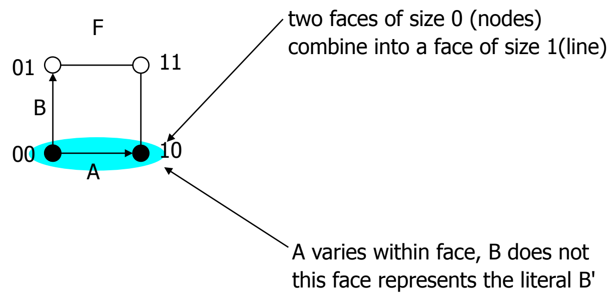
- Visual technique for indentifying when the uniting theorem can be applied
- n input variables = n-dimensional "cube"



Mapping truth tables onto Boolean cubes

- Uniting theorem combines two "faces" of a cube into a larger "face"
- Example:

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

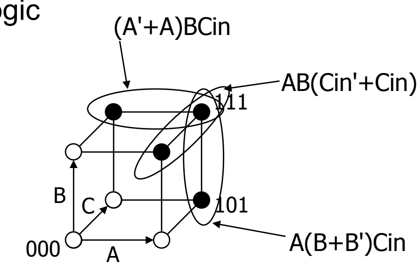


ON-set = solid nodes
OFF-set = empty nodes
DC-set = ×'d nodes

Three variable example

- Binary full-adder carry-out logic

A	B	Cin	Cout
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

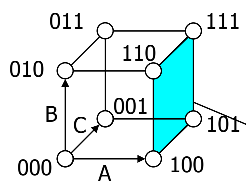


the on-set is completely covered by the combination (OR) of the subcubes of lower dimensionality - note that “111” is covered three times

Cout = BCin+AB+ACin

Higher dimensional cubes

- Sub-cubes of higher dimension than 2



F(A,B,C) = Σm(4,5,6,7)

on-set forms a square
i.e., a cube of dimension 2

represents an expression in one variable
i.e., 3 dimensions – 2 dimensions

A is asserted (true) and unchanged
B and C vary

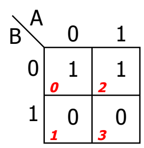
This subcube represents the literal A

m-dimensional cubes in a n-dimensional Boolean space

- In a 3-cube (three variables):
  - a 0-cube, i.e., a single node, yields a term in 3 literals
  - a 1-cube, i.e., a line of two nodes, yields a term in 2 literals
  - a 2-cube, i.e., a plane of four nodes, yields a term in 1 literal
  - a 3-cube, i.e., a cube of eight nodes, yields a constant term "1"
- In general,
  - an m-subcube within an n-cube (m < n) yields a term with n – m literals

Karnaugh maps

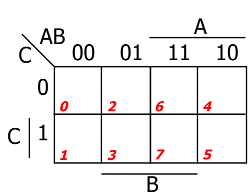
- Flat map of Boolean cube
  - wrap–around at edges
  - hard to draw and visualize for more than 4 dimensions
  - virtually impossible for more than 6 dimensions
- Alternative to truth-tables to help visualize adjacencies
  - guide to applying the uniting theorem
  - on-set elements with only one variable changing value are adjacent unlike the situation in a linear truth-table

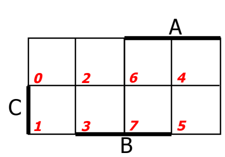


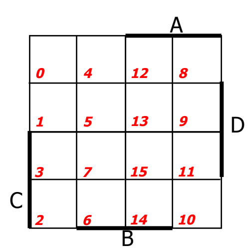
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

Karnaugh maps (cont'd)

- Numbering scheme based on Gray–code
  - e.g., 00, 01, 11, 10
  - only a single bit changes in code for adjacent map cells



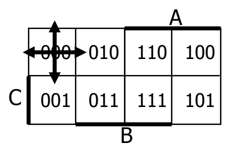


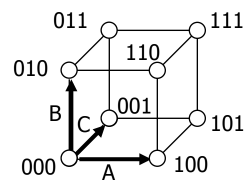


13 = 1101= ABC'D

Adjacencies in Karnaugh maps

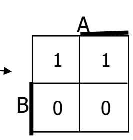
- Wrap from first to last column
- Wrap top row to bottom row

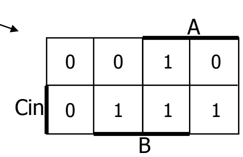


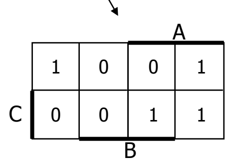


Karnaugh map examples

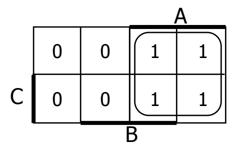
- F =
- Cout =
- f(A,B,C) = Σm(0,4,5,7)



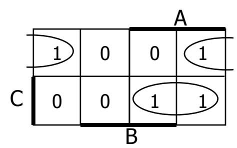




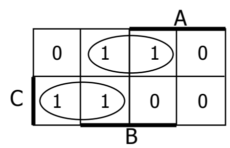
More Karnaugh map examples



G(A,B,C) = A



F(A,B,C) = Σm(0,4,5,7) = AC + B'C'

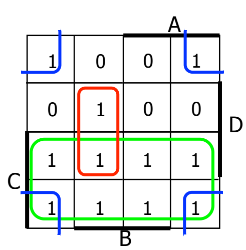


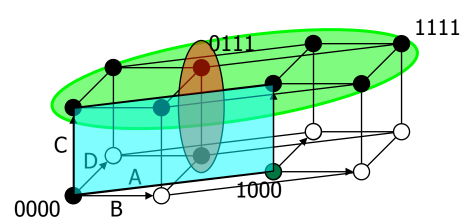
F' simply replace 1's with 0's and vice versa
F'(A,B,C) = Σ m(1,2,3,6)= BC' + A'C

Karnaugh map: 4-variable example

- F(A,B,C,D) = Σm(0,2,3,5,6,7,8,10,11,14,15)

  F = C + A' B D + B' D'

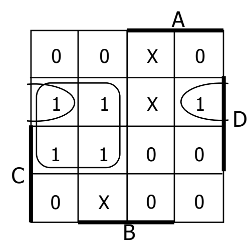




find the smallest number of the largest possible subcubes to cover the ON-set (fewer terms with fewer inputs per term)

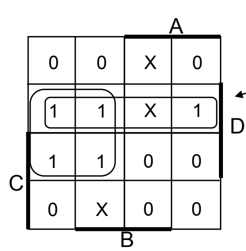
Karnaugh maps: don't cares

- f(A,B,C,D) = Σ m(1,3,5,7,9) + d(6,12,13)
  - without don't cares
    - f = A′D + B′C′D



Karnaugh maps: don't cares (cont'd)

- f(A,B,C,D) = Σ m(1,3,5,7,9) + d(6,12,13)
  - f = A'D + B'C'D without don't cares
  - f = A'D + C'D with don't cares



by using don't care as a "1"
a 2-cube can be formed
rather than a 1-cube to cover
this node

don't cares can be treated as
1s or 0s
depending on which is more
advantageous

Activity

- Minimize the function F = Σ m(0, 2, 7, 8, 14, 15) + d(3, 6, 9, 12, 13)

A

1	0	X	1
0	0	X	X
X	1	1	0
1	X	1	0

D
C
B

Does BC+A'B'D'+B'C'D' = A'C+AB+B'C'D' ?

- NO! Not in general, only if we ignore the cells with don't cares

A

1	0	X	1
0	0	X	X
X	1	1	0
1	X	1	0

D
C
B

F_1 = BC + A'B'D' + B'C'D'

A

1	0	0	1
0	0	0	0
0	1	1	0
1	1	1	0

D
C
B

F_2 = A'C + AB + B'C'D'

A

1	0	1	1
0	0	1	0
1	1	1	0
1	1	1	0

D
C
B

$F_1 \neq F_2$

F_1 + d(3,6,9,12,13) = F_2 + d(3,6,9,12,13)
(don't cares all 1)

F_1 • D(3,6,9,12,13) = F_2 • D(3,6,9,12,13)
(don't cares all 0)

Combinational logic summary (so far)

- Logic functions, truth tables, and switches
  - NOT, AND, OR, NAND, NOR, XOR, . . ., minimal set
- Axioms and theorems of Boolean algebra
  - proofs by re-writing and perfect induction
- Gate logic
  - networks of Boolean functions and their time behavior
- Canonical forms
  - two-level and incompletely specified functions
- Simplification
  - a start at understanding two-level simplification