

Asymptotic Analysis II

CSE 373
Data Structures & Algorithms
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Autumn 2011

Today's Outline

- **Announcements**
 - Midterm dates: Friday October 21 and Friday November 18.
 - Assignment #2, posted soon, due Fri Oct 14 at BEGINNING of lecture
- **Algorithm Analysis**
 - Analyzing code
 - Big-Oh

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Ignoring constant factors

- So **binary search** is $O(\log n)$ and **linear search** is $O(n)$
 - But which is faster?
- Could depend on constant factors:
 - How *many* assignments, additions, etc. for each n
 - E.g. $T(n) = 5,000,000n$ vs. $T(n) = 5n^2$
 - And could depend on size of n (if n is *small* then *constant additive factors could be more important*)
 - E.g. $T(n) = 5,000,000 + \log n$ vs. $T(n) = 10 + n$
- **But** there exists *some* n_0 such that for all $n > n_0$ **binary search** wins
- Let's play with a couple plots to get some intuition...

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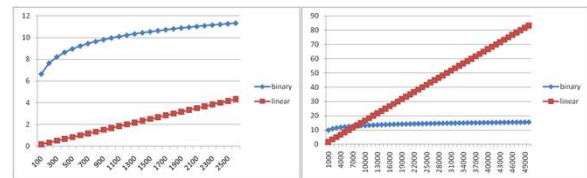
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Linear Search vs. Binary Search

Let's try to "help" **linear search**:

- Run it on a computer 100x as fast (say 2010 model vs. 1990)
 - Use a new compiler/language that is 3x as fast
 - Be a clever programmer to eliminate half the work
 - So doing each iteration is 600x as fast as in binary search
- For small n , **linear search** is faster! But eventually **binary search** wins.



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Asymptotic notation

About to show formal definition of Big-O, which amounts to saying:

1. Eliminate **low-order terms**
2. Eliminate **coefficients**

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log(10n^2)$

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Examples

True or false?

1. $4+3n$ is $O(n)$
2. $n+2\log n$ is $O(\log n)$
3. $\log n + 2$ is $O(1)$
4. n^{50} is $O(1.1^n)$

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Examples

True or false?

- | | |
|-------------------------------|-------|
| 1. $4+3n$ is $O(n)$ | True |
| 2. $n+2\log n$ is $O(\log n)$ | False |
| 3. $\log n+2$ is $O(1)$ | False |
| 4. n^{50} is $O(1.1^n)$ | True |

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Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean the *set of functions with asymptotic behavior less than or equal to $f(n)$*

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) \in O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

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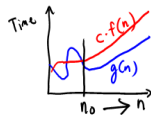
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Formally Big-Oh

Definition: $g(n)$ is in $O(f(n))$ iff there exist positive constants c and n_0 such that

$$g(n) \leq c f(n) \quad \text{for all } n \geq n_0$$



To show $g(n)$ is in $O(f(n))$, pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let $g(n) = 3n^2+17$ and $f(n) = n^2$
 $c = 5$ and $n_0 = 10$ is more than good enough

This is "less than or equal to"

- So $3n^2+17$ is also $O(n^2)$ and $O(2^n)$ etc.

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Using the definition of Big-Oh (Example 1)

Given: $g(n) = 1000n$ & $f(n) = n^2$

Prove: $g(n)$ is in $O(f(n))$

- A valid proof is to find valid c & n_0
- Try: $n_0 = 1000$, $c = 1$
- Also: $n_0 = 1$, $c = 1000$

Def'n:

$g(n)$ is in $O(f(n))$ iff there exist positive constants c and n_0 s.t.
 $g(n) \leq c f(n)$ for all $n \geq n_0$

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Using the definition of Big-Oh (Example 2)

Given: $g(n) = 4n$ & $f(n) = n^2$

Prove: $g(n)$ is in $O(f(n))$

- A valid proof is to find valid c & n_0
- When $n=4$, $g(n)=16$ & $f(n)=16$; this is the crossing over point
- So we can choose $n_0 = 4$, and $c = 1$

- Note: There are many possible choices:
 ex: $n_0 = 78$, and $c = 42$ works fine

Def'n:

$g(n)$ is in $O(f(n))$ iff there exist positive constants c and n_0 s.t.
 $g(n) \leq c f(n)$ for all $n \geq n_0$

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Using the definition of Big-Oh (Example 3)

Given: $g(n) = n^4$ & $f(n) = 2^n$

Prove: $g(n)$ is in $O(f(n))$

- A valid proof is to find valid c & n_0
- One possible answer: $n_0 = 20$, and $c = 1$

Def'n:

$g(n)$ is in $O(f(n))$ iff there exist positive constants c and n_0 s.t.
 $g(n) \leq c f(n)$ for all $n \geq n_0$

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What's with the c ?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called c)
- Consider:

$$g(n) = 7n+5$$

$$f(n) = n$$
- These have the same asymptotic behavior (linear), so $g(n)$ is in $O(f(n))$ even though $g(n)$ is always larger
- There is no positive n_0 such that $g(n) \leq f(n)$ for all $n \geq n_0$
- The ' c ' in the definition allows for that:

$$g(n) \leq c f(n) \text{ for all } n \geq n_0$$
- To prove $g(n)$ is in $O(f(n))$, have $c = 12$, $n_0 = 1$

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Big Oh: Common Categories

From fastest to slowest:

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic ($\log_k n$, $\log n^2$ is $O(\log n)$)
$O(n)$	linear
$O(n \log n)$	" $n \log n$ "
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some $k > 1$ "

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More Definitions

- Upper bound:** $O(f(n))$ is the set of all functions asymptotically less than or equal to $f(n)$
 - $g(n)$ is in $O(f(n))$ if there exist positive constants c and n_0 such that

$$g(n) \leq c f(n) \text{ for all } n \geq n_0$$
- Lower bound:** $\Omega(f(n))$ is the set of all functions asymptotically greater than or equal to $f(n)$
 - $g(n)$ is in $\Omega(f(n))$ if there exist positive constants c and n_0 such that

$$g(n) \geq c f(n) \text{ for all } n \geq n_0$$
- Tight bound:** $\Theta(f(n))$ is the set of all functions asymptotically equal to $f(n)$
 - $g(n)$ is in $\Theta(f(n))$ if **both**: $g(n)$ is in $O(f(n))$ AND $g(n)$ is in $\Omega(f(n))$

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Even More Definitions...

$O(f(n))$ is the set of all functions asymptotically less than or equal to $f(n)$

- $g(n)$ is in $O(f(n))$ if there exist positive constants c and n_0 such that

$$g(n) \leq c f(n) \text{ for all } n \geq n_0$$

$o(f(n))$ is the set of all functions asymptotically less than $f(n)$

- $g(n)$ is in $o(f(n))$ if for any positive constant c , there exists a positive constant n_0 such that

$$g(n) < c f(n) \text{ for all } n \geq n_0$$

$\Omega(f(n))$ is the set of all functions asymptotically greater than or equal to $f(n)$

- $g(n)$ is in $\Omega(f(n))$ if there exist positive constants c and n_0 such that

$$g(n) \geq c f(n) \text{ for all } n \geq n_0$$

$\omega(f(n))$ is the set of all functions asymptotically greater than $f(n)$

- $g(n)$ is in $\omega(f(n))$ if for any positive constant c , there exists a positive constant n_0 such that

$$g(n) > c f(n) \text{ for all } n \geq n_0$$

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Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	\leq
Ω	\geq
Θ	$=$
o	$<$
ω	$>$

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Types of Analysis

Two orthogonal axes:

- bound flavor** (usually we talk about upper or tight)
 - upper bound (O , o)
 - lower bound (Ω , ω)
 - asymptotically tight (Θ)
- analysis case** (usually we talk about worst)
 - worst case (adversary)
 - average case
 - best case
 - "amortized"

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Which Function Grows Faster?

$$n^3 + 2n^2 \text{ vs. } 100n^2 + 1000$$

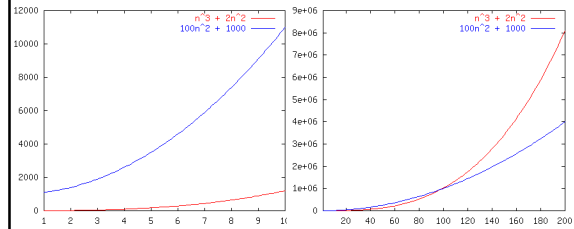
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Which Function Grows Faster?

$$n^3 + 2n^2 \text{ vs. } 100n^2 + 1000$$



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Which Function Grows Faster?

$$n^{0.1} \text{ vs. } \log n$$

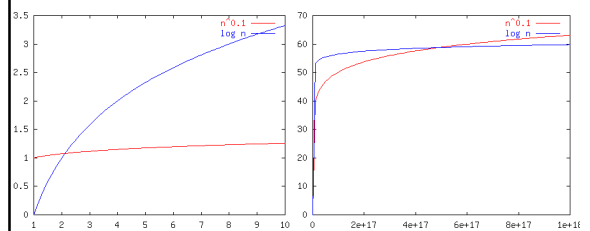
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Which Function Grows Faster?

$$n^{0.1} \text{ vs. } \log n$$



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Which Function Grows Faster?

$$5n^5 \text{ vs. } n!$$

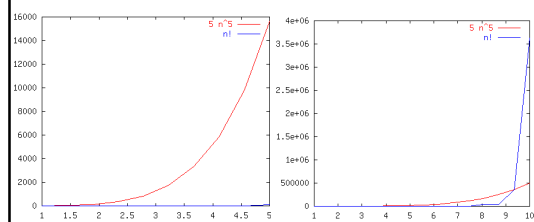
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Which Function Grows Faster?

$$5n^5 \text{ vs. } n!$$



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Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1

for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1
```

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More Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    if (cond) {
      do_stuff(sum)
    } else {
      for k = 1 to n*n
        sum += 1
```

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Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for large n and is independent of any computer / coding trick
 - But you can "abuse" it to be misled about trade-offs
 - Example: $n^{1/10}$ vs. $\log n$
 - Asymptotically $n^{1/10}$ grows more quickly
 - But the "cross-over" point is around $5 * 10^{17}$
 - So if you have input size less than 2^{58} , prefer $n^{1/10}$
- Comparing $O()$ for small n values can be misleading
 - Quicksort: $O(n \log n)$ (expected)
 - Insertion Sort: $O(n^2)$ (expected)
 - Yet in reality Insertion Sort is faster for small n 's
 - We'll learn about these sorts later

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Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
 - Examine the algorithm itself, mathematically, not the implementation
 - Reason about performance as a function of n
 - Be able to mathematically prove things about performance
- Yet, timing has its place
 - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
 - Ex: Benchmarking graphics cards
 - We will do some timing in our homeworks
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful

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