Graphs: Minimum Spanning Trees (Chapter 9)

CSE 373
Data Structures and Algorithms

Today's Outline

- **Admin**:
  - Midterm #2 – Friday Nov 18th, topic list has been posted
  - HW #5 – Graphs, partners allowed, due after Thanksgiving
- **Graphs**
  - Shortest Paths
  - Minimum Spanning Trees

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'**=(**V**, **E'**) such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected

G' is a minimum spanning tree.

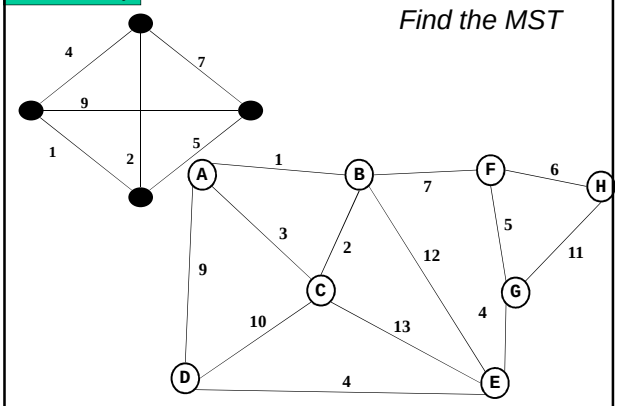
- $\sum_{(u,v)\in E'} C_{uv}$ is minimal

Applications:

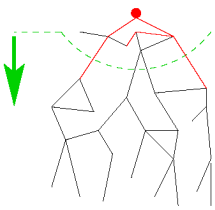
- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

Student Activity

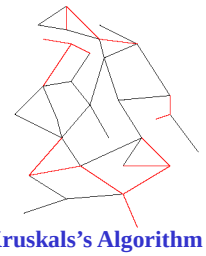
Find the MST



Two Different Approaches



Prim's Algorithm
Almost identical to Dijkstra's



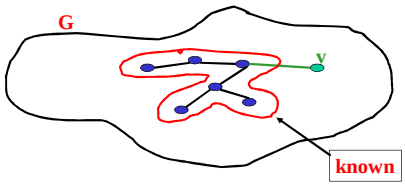
Kruskals's Algorithm
Completely different!

Prim's algorithm

Idea: Grow a tree by picking a vertex from the unknown set that has the smallest cost. Here cost = cost of the edge that connects that vertex to the known set. *Pick the vertex with the smallest cost that connects "known" to "unknown."*

A *node-based* greedy algorithm

Builds MST by greedily adding nodes



Prim's Algorithm vs. Dijkstra's

Recall:

Dijkstra picked the unknown vertex with smallest cost where cost = ***distance to the source***.

Prim's pick the unknown vertex with smallest cost where cost = ***distance from this vertex to the known set*** (in other words, the cost of the smallest edge connecting this vertex to the known set)

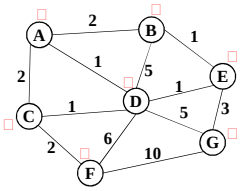
- Otherwise identical
- Compare to slides in Dijkstra lecture!

Prim's Algorithm for MST

1. For each node **v**, set **`v.cost = ∞`** and **`v.known = false`**
2. Choose any node **`v`**. (this is like your "start" vertex in Dijkstra)
   a) Mark **v** as known
   b) For each edge **`(v,u)`** with weight **`w`**: set **`u.cost=w`** and **`u.prev=v`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known and add **`(v, v.prev)`** to output (the MST)
   c) For each edge **`(v,u)`** with weight **`w`**,

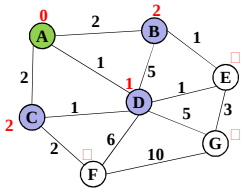
```
if(w < u.cost) {
  u.cost = w;
  u.prev = v;
}
```


Example: Find MST using Prim's



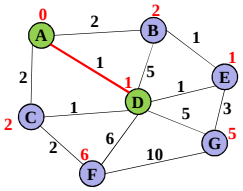
vertex	known?	cost	prev
A		??	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example: Find MST using Prim's



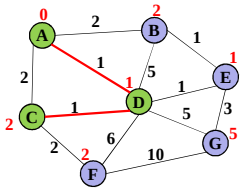
vertex	known?	cost	prev
A	Y	0	
B		2	A
C		2	A
D		1	A
E		??	
F		??	
G		??	

Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B		2	A
C		1	D
D	Y	1	A
E		1	D
F		6	D
G		5	D

Example: Find MST using Prim's



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F		2	C
G		5	D

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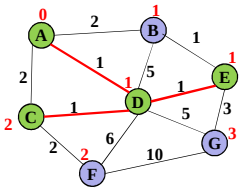
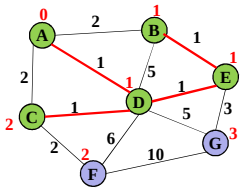


vertex	known?	cost	prev
A	Y	0	
B		1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

Example: Find MST using Prim's



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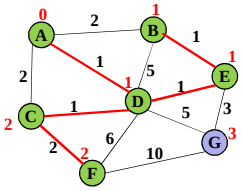

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A	Y	0	
B	Y	1	E
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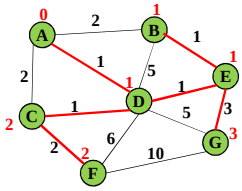
G		3	E

Example: Find MST using Prim's



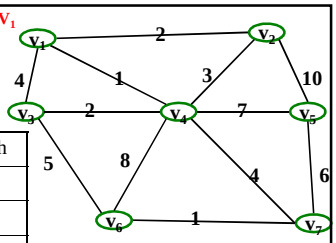
vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G	Y	3	E

Student Activity

Find MST using Prim's

Start with V_1

V	Kwn	Distance	path
v1			
v2			
v3			
v4			
v5			
v6			
v7			



Order Declared Known:

V_1

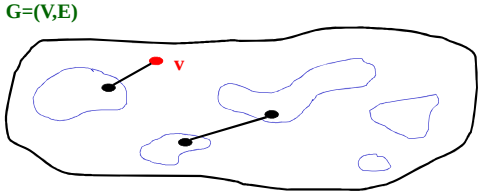
Prim's Analysis

- Correctness ??
  - Intuitively similar to Dijkstra
- Run-time
  - Same as Dijkstra
  - $O(|E| \log |V|)$ using a priority queue

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)



Kruskal's Algorithm for MST

An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge `(u,v)` and mark it
   b. If **u** and **v** are not already connected, add `(u,v)` to the MST and mark `u` and `v` as connected to each other

Kruskal's pseudo code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

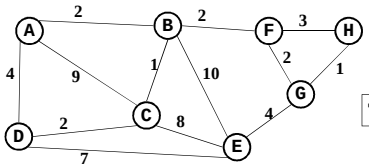
|E| heap ops

2|E| finds

|V| unions

Student Activity

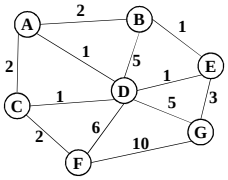
Find MST using Kruskal's



Total Cost:

- **Now find the MST using Prim's method.**
- **Under what conditions will these methods give the same result?**

Example: Find MST using Kruskal's

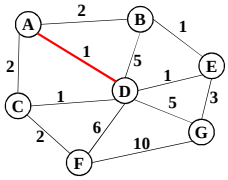


Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output:

Note: At each step, the union/find sets are the trees in the forest

Example: Find MST using Kruskal's



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Output: (A,D)

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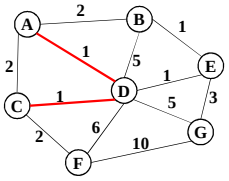

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Output: (A,D), (C,D)

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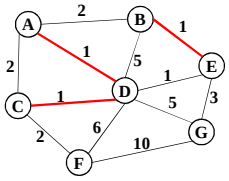

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Output: (A,D), (C,D), (B,E)

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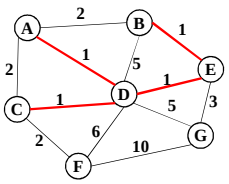
Example: Find MST using Kruskal's



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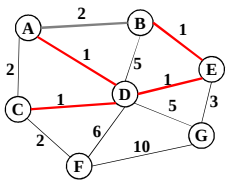

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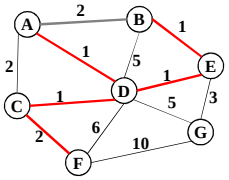

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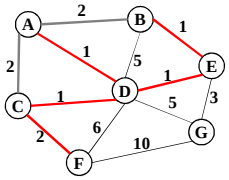

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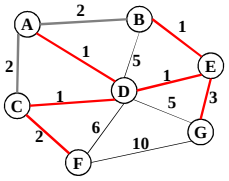

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